

IDEMPOTENT/TROPICAL ANALYSIS, THE HAMILTON-JACOBI AND BELLMAN EQUATIONS

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ABSTRACT. Tropical and idempotent analysis with their relations to the Hamilton-Jacobi and matrix Bellman equations are discussed. Some dequantization procedures are important in tropical and idempotent mathematics. In particular, the Hamilton-Jacobi-Bellman equation is treated as a result of the Maslov dequantization applied to the Schrödinger equation. This leads to a linearity of the Hamilton-Jacobi-Bellman equation over tropical algebras. The correspondence principle and the superposition principle of idempotent mathematics are formulated and examined. The matrix Bellman equation and its applications to optimization problems on graphs are discussed. Universal algorithms for numerical algorithms in idempotent mathematics are investigated. In particular, an idempotent version of interval analysis is briefly discussed.

In dear memory of my beloved wife Irina.

1. INTRODUCTION

In these lecture notes we shall discuss some important problems of tropical and idempotent mathematics and especially those of idempotent and tropical analysis. Relations to the Hamilton-Jacobi and matrix Bellman equations will be examined. Applications of general principles of idempotent mathematics to numerical algorithms and their computer implementations will be discussed.

Tropical mathematics can be treated as a result of a dequantization of the traditional mathematics as the Planck constant tends to zero taking imaginary values. This kind of dequantization is known as the Maslov dequantization and it leads to a mathematics over tropical algebras like the max-plus algebra. The so-called idempotent dequantization is a generalization of the Maslov dequantization. The idempotent dequantization leads to mathematics over idempotent semirings (exact definitions see below in sections 2 and 3). For example, the field of real or complex numbers can

be treated as a quantum object whereas idempotent semirings can be examined as "classical" or "semiclassical" objects (a semiring is called idempotent if the semiring addition is idempotent, i.e. $x \oplus x = x$), see [39–42]. Some other dequantization procedures lead to interesting applications, e.g., to convex geometry, see below and [46, 55, 56].

Tropical algebras are idempotent semirings (and semifields). Thus tropical mathematics is a part of idempotent mathematics. Tropical algebraic geometry can be regarded as a result of the Maslov dequantization applied to the traditional algebraic geometry (O. Viro, G. Mikhalkin), see, e.g., [32, 72, 73, 94–96]. There are interesting relations and applications to the traditional convex geometry.

In the spirit of N.Bohr's correspondence principle there is a (heuristic) correspondence between important, useful, and interesting constructions and results over fields and similar constructions and results over idempotent semirings. A systematic application of this correspondence principle leads to a variety of theoretical and applied results [39–43], see Figure 1.

The history of the subject is discussed, e.g., in [39], with extensive bibliography. See also [15, 17, 18, 20, 22, 40–42, 45].

V.P. Maslov's idempotent superposition principle means that many non-linear problems related to extremal problems are linear over suitable idempotent semirings. The principle is very important for applications including

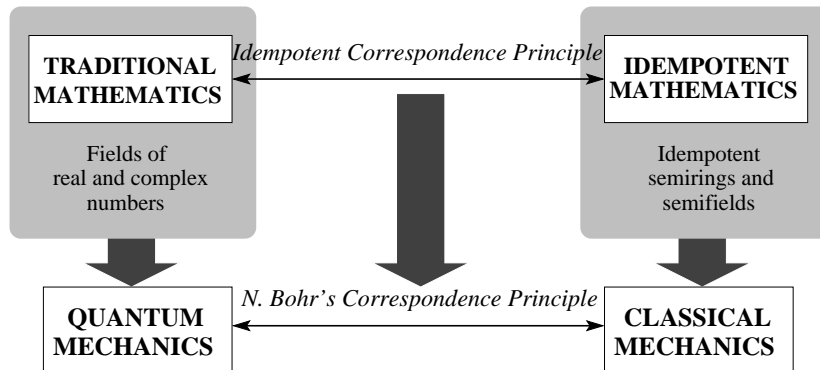


FIGURE 1. Relations between idempotent and traditional mathematics.

numerical and parallel computations. See V.P. Maslov's original formulation in [63–65], as well as [6, 14, 15, 17, 18, 20, 22, 33, 39–43, 45], and below.

2. THE MASLOV DEQUANTIZATION

Let \mathbf{R} and \mathbf{C} be the fields of real and complex numbers. The so-called max-plus algebra $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ is defined by the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

The max-plus algebra can be seen as a result of the *Maslov dequantization* of the semifield \mathbf{R}_+ of all nonnegative numbers with the usual arithmetics. The change of variables

$$x \mapsto u = h \log x,$$

where $h > 0$, defines a map $\Phi_h: \mathbf{R}_+ \rightarrow \mathbf{R} \cup \{-\infty\}$, see Fig. 2. Let the addition and multiplication operations be mapped from \mathbf{R}_+ to $\mathbf{R} \cup \{-\infty\}$ by Φ_h , i.e. let

$$\begin{aligned} u \oplus_h v &= h \log(\exp(u/h) + \exp(v/h)), & u \odot v &= u + v, \\ \mathbf{0} &= -\infty = \Phi_h(0), & \mathbf{1} &= 0 = \Phi_h(1). \end{aligned}$$

It can be easily checked that $u \oplus_h v \rightarrow \max\{u, v\}$ as $h \rightarrow 0$. This deformation of the algebraic structure borrowed from \mathbf{R}_+ brings us to the semifield \mathbf{R}_{\max} , known as the *max-plus algebra*, with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$.

The semifield \mathbf{R}_{\max} is a typical example of an *idempotent semiring*; this is a semiring with idempotent addition, i.e., $x \oplus x = x$ for arbitrary element x of this semiring.

The semifield \mathbf{R}_{\max} is also called a *tropical algebra*. The semifield $\mathbf{R}^{(h)} = \Phi_h(\mathbf{R}_+)$ with operations \oplus_h and \odot (i.e. $+$) is called a *subtropical algebra*.

The semifield $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with operations $\oplus = \min$ and $\odot = +$ ($\mathbf{0} = +\infty, \mathbf{1} = 0$) is isomorphic to \mathbf{R}_{\max} .

The analogy with quantization is obvious; the parameter h plays the role of the Planck constant. The map $x \mapsto |x|$ and the Maslov dequantization

for \mathbf{R}_+ give us a natural transition from the field \mathbf{C} (or \mathbf{R}) to the max-plus algebra \mathbf{R}_{\max} . *We will also call this transition the Maslov dequantization.* In fact the Maslov dequantization corresponds to the usual Schrödinger dequantization but for imaginary values of the Planck constant (see below). The transition from numerical fields to the max-plus algebra \mathbf{R}_{\max} (or similar semifields) in mathematical constructions and results generates the so called *tropical mathematics*. The so-called *idempotent dequantization* is a generalization of the Maslov dequantization; this is the transition from

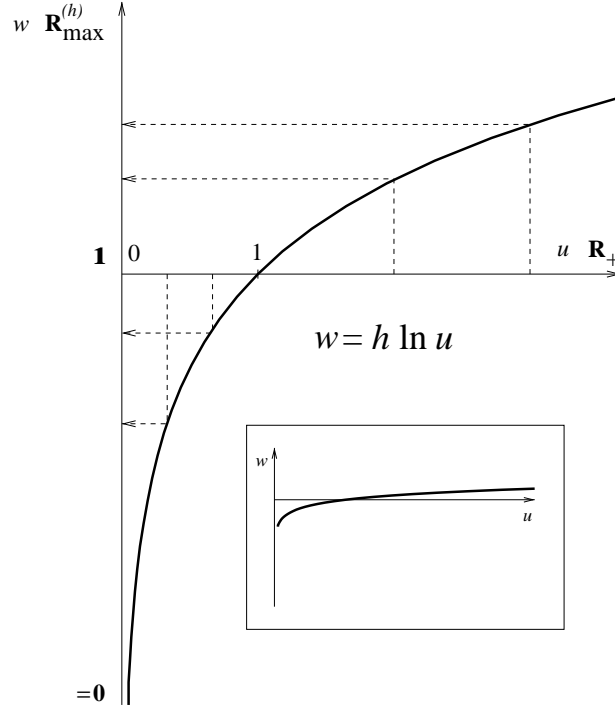


FIGURE 2. Deformation of \mathbf{R}_+ to $\mathbf{R}_{\max}^{(h)}$. Inset: the same for a small value of h .

basic fields to idempotent semirings in mathematical constructions and results without any deformation. The idempotent dequantization generates the so-called *idempotent mathematics*, i.e. mathematics over idempotent semifields and semirings.

Remark. The term 'tropical' appeared in [89] for a discrete version of the max-plus algebra (as a suggestion of Christian Choffrut). On the other hand V.P. Maslov used this term in 80s in his talks and works on economical applications of his idempotent analysis (related to colonial politics). For the most part of modern authors, 'tropical' means 'over \mathbf{R}_{\max} (or \mathbf{R}_{\min})' and tropical algebras are \mathbf{R}_{\max} and \mathbf{R}_{\min} . The terms 'max-plus', 'max-algebra' and 'min-plus' are often used in the same sense.

3. SEMIRINGS AND SEMIFIELDS. THE IDEMPOTENT CORRESPONDENCE PRINCIPLE

Consider a set S equipped with two algebraic operations: *addition* \oplus and *multiplication* \odot . It is a *semiring* if the following conditions are satisfied:

- the addition \oplus and the multiplication \odot are associative;
- the addition \oplus is commutative;
- the multiplication \odot is distributive with respect to the addition \oplus :

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

and

$$(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$$

for all $x, y, z \in S$.

A *unity* (we suppose that it exists) of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* (if it exists) of a semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x$, $\mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}$ for all $x \in S$. A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. A semiring S with neutral element $\mathbf{1}$ is called a *semifield* if every

nonzero element of S is invertible with respect to the multiplication. For the theory of semirings and semifields the reader is referred, e.g., to [26].

The analogy with quantum physics discussed in Section 2 and below leads to the following *idempotent correspondence principle*:

There is a (heuristic) correspondence between important, useful and interesting constructions and results over the field of complex (or real) numbers (or the semifield of nonnegative numbers) and similar constructions and results over idempotent semirings in the spirit of N. Bohr's correspondence principle in quantum theory [40–42].

This principle can be also applied to algorithms and their software and hardware implementations. Examples are discussed below; see also [39–42, 47–50, 53–57]

4. IDEMPOTENT ANALYSIS

Idempotent analysis deals with functions taking their values in an idempotent semiring and the corresponding function spaces. Idempotent analysis was initially constructed by V. P. Maslov and his collaborators and then developed by many authors. The subject is presented in the book of V. N. Kolokoltsov and V. P. Maslov [33] (a version of this book in Russian was published in 1994).

Let S be an arbitrary semiring with idempotent addition \oplus (which is always assumed to be commutative), multiplication \odot , and unit $\mathbf{1}$. The set S is equipped with the *standard partial order* \preceq : by definition, $a \preceq b$ if and only if $a \oplus b = b$. If S contains a zero element $\mathbf{0}$, then all elements of S are nonnegative: $\mathbf{0} \preceq a$ for all $a \in S$. Due to the existence of this order, idempotent analysis is closely related to the lattice theory, theory of vector lattices, and theory of ordered spaces. Moreover, this partial order allows to model a number of basic “topological” concepts and results of idempotent analysis on the purely algebraic level; this line of reasoning was examined systematically in [39]–[57] and [18].

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \rightarrow S$, where X is an arbitrary set and S is an idempotent semiring. Functions with values in S can be added, multiplied by each other, and multiplied by elements of S pointwise.

The idempotent analog of a linear functional space is a set of S -valued functions that is closed under addition of functions and multiplication of functions by elements of S , or an S -semimodule. Consider, e.g., the S -semimodule $B(X, S)$ of all functions $X \rightarrow S$ that are bounded in the sense of the standard order on S .

If $S = \mathbf{R}_{\max}$, then the idempotent analog of integration is defined by the formula

$$(1) \quad I(\varphi) = \int_X^{\oplus} \varphi(x) dx = \sup_{x \in X} \varphi(x),$$

where $\varphi \in B(X, S)$. Indeed, a Riemann sum of the form $\sum_i \varphi(x_i) \cdot \sigma_i$ corresponds to the expression $\bigoplus_i \varphi(x_i) \odot \sigma_i = \max_i \{\varphi(x_i) + \sigma_i\}$, which tends to the right-hand side of (1) as $\sigma_i \rightarrow 0$. Of course, this is a purely heuristic argument.

Formula (1) defines the *idempotent* (or *Maslov*) *integral* not only for functions taking values in \mathbf{R}_{\max} , but also in the general case when any of bounded (from above) subsets of S has the least upper bound.

An *idempotent* (or *Maslov*) *measure* on X is defined by the formula $m_\psi(Y) = \sup_{x \in Y} \psi(x)$, where $\psi \in B(X, S)$ is a fixed function. The integral with respect to this measure is defined by the formula

$$(2) \quad I_\psi(\varphi) = \int_X^{\oplus} \varphi(x) dm_\psi = \int_X^{\oplus} \varphi(x) \odot \psi(x) dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)).$$

Obviously, if $S = \mathbf{R}_{\min}$, then the standard order is opposite to the conventional order \leq , so in this case equation (2) takes the form

$$\int_X^{\oplus} \varphi(x) dm_{\psi} = \int_X^{\oplus} \varphi(x) \odot \psi(x) dx = \inf_{x \in X} (\varphi(x) \odot \psi(x)),$$

where \inf is understood in the sense of the conventional order \leq .

We shall see that in idempotent analysis measures and generalized functions (versions of distributions in the sense of L. Schwartz) are generated by usual functions. For example the δ -functional $\delta_y : \varphi(\cdot) \mapsto \varphi(y)$ is generated by the function

$$\delta_y(x) = \begin{cases} \mathbf{1}, & \text{if } x = y, \\ \mathbf{0}, & \text{if } x \neq y. \end{cases}$$

It is clear that

$$\varphi(y) = \int_X^{\oplus} \delta_y(x) \odot \varphi(x) dx = \sup_x (\delta_y(x) \odot \varphi(x))$$

.

5. THE SUPERPOSITION PRINCIPLE AND LINEAR EQUATIONS

5.1. Heuristics. Basic equations of quantum theory are linear; this is the superposition principle in quantum mechanics. The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However, it is linear over the semirings \mathbf{R}_{\max} and \mathbf{R}_{\min} . Similarly, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings; this is V. P. Maslov’s idempotent superposition principle, see [63–65]. More generally, the idempotent superposition principle means that although some important problems and equations (related to extremal problems, e.g., optimization problems, the Bellman equation and its instances, the Hamilton–Jacobi equation) are nonlinear in the usual sense, they can be treated as linear over appropriate idempotent semirings. For instance, the finite-dimensional stationary Bellman equation can be written in the

form $X = H \odot X \oplus F$, where X , H , F are matrices with coefficients in an idempotent semiring S and the unknown matrix X is determined by H and F , see below and [6, 14, 15, 20, 22, 28, 29]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbf{R}_{\max}$ and $S = \mathbf{R}_{\min}$, respectively. It is known that principal optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman's shortest path algorithm corresponds to a version of Jacobi's algorithm, Ford's algorithm corresponds to the Gauss–Seidel iterative scheme, etc. [14, 15].

The linearity of the Hamilton–Jacobi equation over \mathbf{R}_{\min} and \mathbf{R}_{\max} , which is the result of the Maslov dequantization of the Schrödinger equation, is closely related to the (conventional) linearity of the Schrödinger equation and can be deduced from this linearity. Thus, it is possible to borrow standard ideas and methods of linear analysis and apply them to a new area.

Consider a classical dynamical system specified by the Hamiltonian

$$H = H(p, x) = \sum_{i=1}^N \frac{p_i^2}{2m_i} + V(x),$$

where $x = (x_1, \dots, x_N)$ are generalized coordinates, $p = (p_1, \dots, p_N)$ are generalized momenta, m_i are generalized masses, and $V(x)$ is the potential. In this case the Lagrangian $L(x, \dot{x}, t)$ has the form

$$L(x, \dot{x}, t) = \sum_{i=1}^N m_i \frac{\dot{x}_i^2}{2} - V(x),$$

where $\dot{x} = (\dot{x}_1, \dots, \dot{x}_N)$, $\dot{x}_i = dx_i/dt$. The value function $S(x, t)$ of the action functional has the form

$$S = \int_{t_0}^t L(x(t), \dot{x}(t), t) dt,$$

where the integration is performed along the actual trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle, or the least action principle).

For fixed values of t and t_0 and arbitrary trajectories $x(t)$, the action functional $S = S(x(t))$ can be considered as a function taking the set of curves (trajectories) to the set of real numbers which can be treated as elements of \mathbf{R}_{\min} . In this case the minimum of the action functional can be viewed as the Maslov integral of this function over the set of trajectories or an idempotent analog of the Euclidean version of the Feynman path integral. The minimum of the action functional corresponds to the maximum of e^{-S} , i.e. idempotent integral $\int_{\{paths\}}^{\oplus} e^{-S(x(t))} D\{x(t)\}$ with respect to the max-plus algebra \mathbf{R}_{\max} . Thus the least action principle can be considered as an idempotent version of the well-known Feynman approach to quantum mechanics. The representation of a solution to the Schrödinger equation in terms of the Feynman integral corresponds to the Lax–Oleĭnik solution formula for the Hamilton–Jacobi equation.

Since $\partial S/\partial x_i = p_i$, $\partial S/\partial t = -H(p, x)$, the following Hamilton–Jacobi equation holds:

$$(3) \quad \frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x_i}, x_i\right) = 0.$$

Quantization leads to the Schrödinger equation

$$(4) \quad -\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \hat{H}\psi = H(\hat{p}_i, \hat{x}_i)\psi,$$

where $\psi = \psi(x, t)$ is the wave function, i.e., a time-dependent element of the Hilbert space $L^2(\mathbf{R}^N)$, and \hat{H} is the energy operator obtained by substitution of the momentum operators $\hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial x_i}$ and the coordinate operators $\hat{x}_i: \psi \mapsto x_i \psi$ for the variables p_i and x_i in the Hamiltonian function, respectively. This equation is linear in the conventional sense (the quantum superposition principle). The standard procedure of limit transition

from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function: $\psi(x, t) = a(x, t)e^{iS(x, t)/\hbar}$, and to keep only the leading order as $\hbar \rightarrow 0$ (the ‘semiclassical’ limit).

Instead of doing this, we switch to imaginary values of the Planck constant \hbar by the substitution $h = i\hbar$, assuming $h > 0$. Then the Schrödinger equation (4) becomes similar to the heat equation:

$$(5) \quad h \frac{\partial u}{\partial t} = H \left(-h \frac{\partial}{\partial x_i}, \hat{x}_i \right) u,$$

where the real-valued function u corresponds to the wave function ψ . A similar idea (a switch to imaginary time) is used in the Euclidean quantum field theory; let us remember that time and energy are dual quantities.

Linearity of equation (4) implies linearity of equation (5). Thus if u_1 and u_2 are solutions of (5), then so is their linear combination

$$(6) \quad u = \lambda_1 u_1 + \lambda_2 u_2.$$

Let $S = h \ln u$ or $u = e^{S/h}$ as in Section 2 above. It can easily be checked that equation (5) thus turns to

$$(7) \quad \frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^N \frac{1}{2m_i} \left(\frac{\partial S}{\partial x_i} \right)^2 + h \sum_{i=1}^n \frac{1}{2m_i} \frac{\partial^2 S}{\partial x_i^2}.$$

Thus we have a transition from (3) to (7) by means of the change of variables $\psi = e^{S/h}$. Note that $|\psi| = e^{\operatorname{Re} S/h}$, where $\operatorname{Re} S$ is the real part of S . Now let us consider S as a real variable. The equation (7) is nonlinear in the conventional sense. However, if S_1 and S_2 are its solutions, then so is the function

$$(8) \quad S = \lambda_1 \odot S_1 \oplus_h \lambda_2 \odot S_2$$

obtained from (6) by means of the substitution $S = h \ln u$. Here the generalized multiplication \odot coincides with the ordinary addition and the generalized addition \oplus_h is the image of the conventional addition under the above change of variables. As $h \rightarrow 0$, we obtain the operations of the idempotent semiring \mathbf{R}_{\max} , i.e., $\oplus = \max$ and $\odot = +$, and equation (7)

becomes the Hamilton–Jacobi equation (3), since the third term in the right-hand side of equation (7) vanishes.

Thus it is natural to consider the limit function $S = \lambda_1 \odot S_1 \oplus \lambda_2 \odot S_2$ as a solution of the Hamilton–Jacobi equation and to expect that this equation can be treated as linear over \mathbf{R}_{\max} . This argument (clearly, a heuristic one) can be extended to equations of a more general form. For a rigorous treatment of (semiring) linearity for these equations see, e.g., [33, 43, 85]. Notice that if h is changed to $-h$, then we have that the resulting Hamilton–Jacobi equation is linear over \mathbf{R}_{\min} .

The idempotent superposition principle indicates that there exist important nonlinear (in the traditional sense) problems that are linear over idempotent semirings. The idempotent linear functional analysis (see below) is a natural tool for investigation of those nonlinear infinite-dimensional problems that possess this property.

5.2. The Cauchy problem for the Hamilton–Jacobi equations. A rigorous “idempotent” approach to the investigation of the Hamilton–Jacobi equation was developed by V.N. Kolokoltsov and V.P. Maslov [33] (a Russian version of this book was published in 1994); see also [71, 85, 92, 93].

Let us consider, inspired by a long tradition, the well-known Cauchy problem for the Hamilton–Jacobi equation (3). Given the action function at time T

$$(9) \quad S(T, x) = S_T(x) = \varphi(x), \quad x \in \mathbf{R}^N,$$

the Cauchy problem asks to reconstruct $S(t, x)$ for $x \in \mathbf{R}^N$ during the time interval $0 \leq t \leq T$.

We shall discuss the min-plus linearity of this problem and denote by U_t the resolving operator, i.e. the map which assigns to each given $S_T(x)$ the solution $S(t, x)$ of the Cauchy problem in the interval $0 \leq t \leq T$. Then the map U_t , for each t , is a linear (over \mathbf{R}_{\min}) operator in the space $\text{LSC}(\mathbf{R}^n, \mathbf{R}_{\min})$ of lower semicontinuous functions taking their values in

\mathbf{R}_{\min} . Moreover U_t is an integral operator (in the sense of idempotent mathematics) of the form:

$$(10) \quad (U_t \varphi)(x) = \int^{\oplus} \varphi(y) K_t(x, y) dy = \inf_y \{ \varphi(y) + K_t(x, y) \},$$

where $K_t(x, y)$, as a function of $y \in \mathbf{R}^n$, is bounded from below and lower semicontinuous. See [33, 85] for details.

The operator U_t (as well as other integral operators, see Section 7 below) has the following property:

$$(11) \quad U_t \left(\bigoplus_{\nu} \varphi_{\nu} \right) = \bigoplus_{\nu} (U_t \varphi_{\nu}),$$

where $\{\varphi_{\nu}\}$ is a bounded set of elements in $\text{LSC}(\mathbf{R}^n, \mathbf{R}_{\min})$. So if we have such a family of functions $S_{\nu}(T, x)$ and $S(T, x) = \int^{\oplus} S_{\nu}(T, x) d\nu = \inf_{\nu} (S_{\nu}(T, x))$, then the solution of the Cauchy problem is expressed as $S(t, x) = \inf_{\nu} (S_{\nu}(t, x))$.

Relations between the “idempotent approach”, viscosity solutions and minimax solutions in the sense of A.I. Subbotin [92, 93] are examined, e.g., in [85] in details; see also W.M. McEneaney [71]. To this end, let us mention that more general Hamiltonians of the form $H = H(t, x, p)$ (satisfying some additional conditions) and different kinds of solution spaces are also considered in the literature.

The situation is similar for the Cauchy problem for the homogeneous Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x}\right) = 0, \quad S_{t=0} = S_0(x),$$

where $H : \mathbf{R}^n \mapsto \mathbf{R}$ is a convex (not strictly) first order homogeneous function

$$H(p) = \sup_{(f, g) \in V} (f \cdot p + g), \quad f \in \mathbf{R}^n, \quad g \in \mathbf{R},$$

and V is a compact set in \mathbf{R}^{n+1} . See [33].

To develop a rigorous “idempotent” approach to differential equations and other problems, one needs an idempotent version of analysis and, especially, functional analysis. See Section 7 below.

6. CONVOLUTION AND THE FOURIER–LEGENDRE TRANSFORM

Let G be a group. Then the space $\mathcal{B}(G, \mathbf{R}_{\max})$ of all bounded functions $G \rightarrow \mathbf{R}_{\max}$ (see above) is an idempotent semiring with respect to the following analog \circledast of the usual convolution:

$$(\varphi(x) \circledast \psi)(g) = \int_G^{\oplus} \varphi(x) \odot \psi(x^{-1} \cdot g) dx = \sup_{x \in G} (\varphi(x) + \psi(x^{-1} \cdot g)).$$

Of course, it is possible to consider other “function spaces” (and other basic semirings instead of \mathbf{R}_{\max}).

Let $G = \mathbf{R}^n$, where \mathbf{R}^n is considered as a topological group with respect to the vector addition. The conventional Fourier–Laplace transform is defined as

$$(12) \quad \varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x) dx$$

where $e^{i\xi \cdot x}$ is a character of the group G , i.e., a solution of the following functional equation:

$$f(x + y) = f(x)f(y).$$

The idempotent analog of this equation is

$$f(x + y) = f(x) \odot f(y) = f(x) + f(y),$$

so “continuous idempotent characters” are linear functionals of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$. As a result, the transform in (12) assumes the form

$$(13) \quad \varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G^{\oplus} \xi \cdot x \odot \varphi(x) dx = \sup_{x \in G} (\xi \cdot x + \varphi(x)).$$

The transform in (13) is the *Legendre transform* (up to some change of notation) [65]; transforms of this kind establish the correspondence between the Lagrangian and the Hamiltonian formulations of classical mechanics.

The Legendre transform generates an idempotent version of harmonic analysis for the space of convex functions, see, e.g., [61].

Of course, this construction can be generalized to different classes of groups and semirings. Transformations of this type convert the generalized convolution \circledast to the pointwise (generalized) multiplication and possess analogs of some important properties of the usual Fourier transform.

The examples discussed in this sections can be treated as fragments of an idempotent version of the representation theory, see, e.g., [50]. In particular, “idempotent” representations of groups can be examined as representations of the corresponding convolution semirings (i.e. idempotent group semirings) in semimodules.

7. IDEMPOTENT FUNCTIONAL ANALYSIS

Many other idempotent analogs may be given, in particular, for basic constructions and theorems of functional analysis. Idempotent functional analysis is an abstract version of idempotent analysis. For the sake of simplicity take $S = \mathbf{R}_{\max}$ and let X be an arbitrary set. The idempotent integration can be defined by the formula (1), see above. The functional $I(\varphi)$ is linear over S and its values correspond to limiting values of the corresponding analogs of Lebesgue (or Riemann) sums. An idempotent scalar product of functions φ and ψ is defined by the formula

$$\langle \varphi, \psi \rangle = \int_X^{\oplus} \varphi(x) \odot \psi(x) dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)).$$

So it is natural to construct idempotent analogs of integral operators in the form

$$(14) \quad \varphi(y) \mapsto (K\varphi)(x) = \int_Y^{\oplus} K(x, y) \odot \varphi(y) dy = \sup_{y \in Y} \{K(x, y) + \varphi(y)\},$$

where $\varphi(y)$ is an element of a space of functions defined on a set Y , and $K(x, y)$ is an S -valued function on $X \times Y$. Of course, expressions of this type are standard in optimization problems.

Recall that the definitions and constructions described above can be extended to the case of idempotent semirings which are conditionally complete in the sense of the standard order. Using the Maslov integration, one can construct various function spaces as well as idempotent versions of the theory of generalized functions (distributions). For some concrete idempotent function spaces it was proved that every ‘good’ linear operator (in the idempotent sense) can be presented in the form (14); this is an idempotent version of the kernel theorem of L. Schwartz; results of this type were proved by V. N. Kolokoltsov, P. S. Dudnikov and S. N. Samborskii, I. Singer, M. A. Shubin and others. So every ‘good’ linear functional can be presented in the form $\varphi \mapsto \langle \varphi, \psi \rangle$, where \langle, \rangle is an idempotent scalar product.

In the framework of idempotent functional analysis results of this type can be proved in a very general situation. In [47–50, 54, 57] an algebraic version of the idempotent functional analysis is developed; this means that basic (topological) notions and results are simulated in purely algebraic terms (see below). The treatment covers the subject from basic concepts and results (e.g., idempotent analogs of the well-known theorems of Hahn-Banach, Riesz, and Riesz-Fisher) to idempotent analogs of A. Grothendieck’s concepts and results on topological tensor products, nuclear spaces and operators. Abstract idempotent versions of the kernel theorem are formulated. Note that the transition from the usual theory to idempotent functional analysis may be very nontrivial; for example, there are many non-isomorphic idempotent Hilbert spaces. Important results on idempotent functional analysis (duality and separation theorems) were obtained by G. Cohen, S. Gaubert, and J.-P. Quadrat. Idempotent functional analysis has received much attention in the last years, see, e.g., [3, 18, 28–30, 68, 88], [33]–[57] and works cited in [39]. All the results presented in this section are proved in [49] (Subsections 7.1 – 7.4) and in [57] (Subsections 7.5 – 7.10)

7.1. Idempotent semimodules and idempotent linear spaces. An additive semigroup S with commutative addition \oplus is called an *idempotent semigroup* if the relation $x \oplus x = x$ is fulfilled for all elements $x \in S$. If S contains a neutral element, this element is denoted by the symbol $\mathbf{0}$. Any idempotent semigroup is a partially ordered set with respect to the following standard order: $x \preceq y$ if and only if $x \oplus y = y$. It is obvious that this order is well defined and $x \oplus y = \sup\{x, y\}$. Thus, any idempotent semigroup is an upper semilattice; moreover, the concepts of idempotent semigroup and upper semilattice coincide, see [10]. An idempotent semigroup S is called *a-complete* (or *algebraically complete*) if it is complete as an ordered set, i.e., if any subset X in S has the least upper bound $\sup(X)$ denoted by $\oplus X$ and the greatest lower bound $\inf(X)$ denoted by $\wedge X$. This semigroup is called *b-complete* (or *boundedly complete*), if any bounded above subset X of this semigroup (including the empty subset) has the least upper bound $\oplus X$ (in this case, any nonempty subset Y in S has the greatest lower bound $\wedge Y$ and S is a lattice). Note that any *a-complete* or *b-complete* idempotent semiring has the zero element $\mathbf{0}$ that coincides with $\oplus \emptyset$, where \emptyset is the empty set. Certainly, *a-completeness* implies the *b-completeness*. Completion by means of cuts [10] yields an embedding $S \rightarrow \widehat{S}$ of an arbitrary idempotent semigroup S into an *a-complete* idempotent semigroup \widehat{S} (which is called a *normal completion* of S); in addition, $\widehat{\widehat{S}} = S$. The *b-completion* procedure $S \rightarrow \widehat{S}_b$ is defined similarly: if $S \ni \infty = \sup S$, then $\widehat{S}_b = \widehat{S}$; otherwise, $\widehat{S} = \widehat{S}_b \cup \{\infty\}$. An arbitrary *b-complete* idempotent semigroup S also may differ from \widehat{S} only by the element $\infty = \sup S$.

Let S and T be *b-complete* idempotent semigroups. Then, a homomorphism $f : S \rightarrow T$ is said to be a *b-homomorphism* if $f(\oplus X) = \oplus f(X)$ for any bounded subset X in S . If the *b-homomorphism* f is extended to a homomorphism $\widehat{S} \rightarrow \widehat{T}$ of the corresponding normal completions and $f(\oplus X) = \oplus f(X)$ for all $X \subset S$, then f is said to be an *a-homomorphism*.

An idempotent semigroup S equipped with a topology such that the set $\{s \in S \mid s \preceq b\}$ is closed in this topology for any $b \in S$ is called a *topological idempotent semigroup* S .

Proposition 1. *Let S be an a -complete topological idempotent semigroup and T be a b -complete topological idempotent semigroup such that, for any nonempty subsemigroup X in T , the element $\oplus X$ is contained in the topological closure of X in T . Then, a homomorphism $f : T \rightarrow S$ that maps zero into zero is an a -homomorphism if and only if the mapping f is lower semicontinuous in the sense that the set $\{t \in T \mid f(t) \preceq s\}$ is closed in T for any $s \in S$.*

An idempotent semiring K is called *a -complete* (respectively *b -complete*) if K is an a -complete (respectively b -complete) idempotent semigroup and, for any subset (respectively, for any bounded subset) X in K and any $k \in K$, the generalized distributive laws $k \odot (\oplus X) = \oplus (k \odot X)$ and $(\oplus X) \odot k = \oplus (X \odot k)$ are fulfilled. Generalized distributivity implies that any a -complete or b -complete idempotent semiring has a zero element that coincides with $\oplus \emptyset$, where \emptyset is the empty set.

The set $\mathbf{R}(\max, +)$ of real numbers equipped with the idempotent addition $\oplus = \max$ and multiplication $\odot = +$ is an idempotent semiring; in this case, $\mathbf{1} = 0$. Adding the element $\mathbf{0} = -\infty$ to this semiring, we obtain a b -complete semiring $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ with the same operations and the zero element. Adding the element $+\infty$ to \mathbf{R}_{\max} and assuming that $\mathbf{0} \odot (+\infty) = \mathbf{0}$ and $x \odot (+\infty) = +\infty$ for $x \neq \mathbf{0}$ and $x \oplus (+\infty) = +\infty$ for any x , we obtain the a -complete idempotent semiring $\widehat{\mathbf{R}}_{\max} = \mathbf{R}_{\max} \cup \{+\infty\}$. The standard order on $\mathbf{R}(\max, +)$, \mathbf{R}_{\max} and $\widehat{\mathbf{R}}_{\max}$ coincides with the ordinary order. The semirings $\mathbf{R}(\max, +)$ and \mathbf{R}_{\max} are semifields. On the contrary, an a -complete semiring that does not coincide with $\{\mathbf{0}, \mathbf{1}\}$ cannot be a semifield. An important class of examples is related to (topological) vector lattices (see, for example, [10] and [86], Chapter 5). Defining the

sum $x \oplus y$ as $\sup\{x, y\}$ and the multiplication \odot as the addition of vectors, we can interpret the vector lattices as idempotent semifields. Adding the zero element $\mathbf{0}$ to a complete vector lattice (in the sense of [10, 86]), we obtain a b -complete semifield. If, in addition, we add the infinite element, we obtain an a -complete idempotent semiring (which, as an ordered set, coincides with the normal completion of the original lattice).

Important definitions. Let V be an idempotent semigroup and K be an idempotent semiring. Suppose that a multiplication $k, x \mapsto k \odot x$ of all elements from K by the elements from V is defined; moreover, this multiplication is associative and distributive with respect to the addition in V and $\mathbf{1} \odot x = x$, $\mathbf{0} \odot x = \mathbf{0}$ for all $x \in V$. In this case, the semigroup V is called an *idempotent semimodule* (or simply, a *semimodule*) over K . The element $\mathbf{0}_V \in V$ is called the *zero* of the semimodule V if $k \odot \mathbf{0}_V = \mathbf{0}_V$ and $\mathbf{0}_V \oplus x = x$ for any $k \in K$ and $x \in V$. Let V be a semimodule over a b -complete idempotent semiring K . This semimodule is called *b-complete* if it is b -complete as an idempotent semiring and, for any bounded subsets Q in K and X in V , the generalized distributive laws $(\oplus Q) \odot x = \oplus (Q \odot x)$ and $k \odot (\oplus X) = \oplus (k \odot X)$ are fulfilled for all $k \in K$ and $x \in X$. This semimodule is called *a-complete* if it is b -complete and contains the element $\infty = \sup V$.

A semimodule V over a b -complete semifield K is said to be an *idempotent a-space* (*b-space*) if this semimodule is a -complete (respectively, b -complete) and the equality $(\wedge Q) \odot x = \wedge (Q \odot x)$ holds for any nonempty subset Q in K and any $x \in V$, $x \neq \infty = \sup V$. The normal completion \widehat{V} of a b -space V (as an idempotent semigroup) has the structure of an idempotent a -space (and may differ from V only by the element $\infty = \sup V$).

Let V and W be idempotent semimodules over an idempotent semiring K . A mapping $p : V \rightarrow W$ is said to be *linear* (over K) if

$$p(x \oplus y) = p(x) \oplus p(y) \text{ and } p(k \odot x) = k \odot p(x)$$

for any $x, y \in V$ and $k \in K$. Let the semimodules V and W be b -complete. A linear mapping $p : V \rightarrow W$ is said to be b -linear if it is a b -homomorphism of the idempotent semigroup; this mapping is said to be a -linear if it can be extended to an a -homomorphism of the normal completions \widehat{V} and \widehat{W} . Proposition 7.1 (see above) shows that a -linearity simulates (semi)continuity for linear mappings. The normal completion \widehat{K} of the semifield K is a semimodule over K . If $W = \widehat{K}$, then the linear mapping p is called a *linear functional*.

Linear, a -linear and b -linear mappings are also called *linear*, *a -linear* and *b -linear operators* respectively.

Examples of idempotent semimodules and spaces that are the most important for analysis are either subsemimodules of topological vector lattices [86] (or coincide with them) or are dual to them, i.e., consist of linear functionals subject to some regularity condition, for example, consist of a -linear functionals. Concrete examples of idempotent semimodules and spaces of functions (including spaces of bounded, continuous, semicontinuous, convex, concave and Lipschitz functions) see in [33, 48, 49, 57] and below.

7.2. Basic results. Let V be an idempotent b -space over a b -complete semifield K , $x \in \widehat{V}$. Denote by x^* the functional $V \rightarrow \widehat{K}$ defined by the formula $x^*(y) = \wedge \{k \in K \mid y \preceq k \odot x\}$, where y is an arbitrary fixed element from V .

Theorem 2. *For any $x \in \widehat{V}$ the functional x^* is a -linear. Any nonzero a -linear functional f on V is given by $f = x^*$ for a unique suitable element $x \in V$. If $K \neq \{0, 1\}$, then $x = \oplus \{y \in V \mid f(y) \preceq 1\}$.*

Note that results of this type obtained earlier concerning the structure of linear functionals cannot be carried over to subspaces and subsemimodules.

A subsemigroup W in V closed with respect to the multiplication by an arbitrary element from K is called a *b -subspace* in V if the imbedding

$W \rightarrow V$ can be extended to a b -linear mapping. The following result is obtained from Theorem 2 and is the idempotent version of the Hahn–Banach theorem.

Theorem 3. *Any a -linear functional defined on a b -subspace W in V can be extended to an a -linear functional on V . If $x, y \in V$ and $x \neq y$, then there exists an a -linear functional f on V that separates the elements x and y , i.e., $f(x) \neq f(y)$.*

The following statements are easily derived from the definitions and can be regarded as the analogs of the well-known results of the traditional functional analysis (the Banach–Steinhaus and the closed-graph theorems).

Proposition 4. *Suppose that P is a family of a -linear mappings of an a -space V into an a -space W and the mapping $p : V \rightarrow W$ is the pointwise sum of the mappings of this family, i.e., $p(x) = \sup\{p_\alpha(x) | p_\alpha \in P\}$. Then the mapping p is a -linear.*

Proposition 5. *Let V and W be a -spaces. A linear mapping $p : V \rightarrow W$ is a -linear if and only if its graph Γ in $V \times W$ is closed with respect to passing to sums (i.e., to least upper bounds) of its arbitrary subsets.*

In [18] the basic results were generalized for the case of semimodules over the so-called reflexive b -complete semirings.

7.3. Idempotent b -semialgebras. Let K be a b -complete semifield and A be an idempotent b -space over K equipped with the structure of a semiring compatible with the multiplication $K \times A \rightarrow A$ so that the associativity of the multiplication is preserved. In this case, A is called an *idempotent b -semialgebra* over K .

Proposition 6. *For any invertible element $x \in A$ from the b -semialgebra A and any element $y \in A$, the equality $x^*(y) = \mathbf{1}^*(y \odot x^{-1})$ holds, where $\mathbf{1} \in A$.*

The mapping $A \times A \rightarrow \widehat{K}$ defined by the formula $(x, y) \mapsto \langle x, y \rangle = \mathbf{1}^*(x \odot y)$ is called the *canonical scalar product* (or simply *scalar product*). The basic properties of the scalar product are easily derived from Proposition 6 (in particular, the scalar product is commutative if the b -semialgebra A is commutative). The following theorem is an idempotent version of the Riesz–Fisher theorem.

Theorem 7. *Let a b -semialgebra A be a semifield. Then any nonzero a -linear functional f on A can be represented as $f(y) = \langle y, x \rangle$, where $x \in A$, $x \neq \mathbf{0}$ and $\langle \cdot, \cdot \rangle$ is the canonical scalar product on A .*

Remark 8. Using the completion procedures, one can extend all the results obtained to the case of incomplete semirings, spaces, and semimodules, see [49].

Example 9. Let $\mathcal{B}(X)$ be a set of all bounded functions with values belonging to $\mathbf{R}(\max, +)$ on an arbitrary set X and let $\widehat{\mathcal{B}}(X) = \mathcal{B}(X) \cup \{\mathbf{0}\}$. The pointwise idempotent addition of functions $(\varphi_1 \oplus \varphi_2)(x) = \varphi_1(x) \oplus \varphi_2(x)$ and the multiplication $(\varphi_1 \odot \varphi_2)(x) = (\varphi_1(x)) \odot (\varphi_2(x))$ define on $\widehat{\mathcal{B}}(X)$ the structure of a b -semialgebra over the b -complete semifield \mathbf{R}_{\max} . In this case, $\mathbf{1}^*(\varphi) = \sup_{x \in X} \varphi(x)$ and the scalar product is expressed in terms of idempotent integration: $\langle \varphi_1, \varphi_2 \rangle = \sup_{x \in X} (\varphi_1(x) \odot \varphi_2(x)) = \sup_{x \in X} (\varphi_1(x) + \varphi_2(x)) = \int_X^{\oplus} (\varphi_1(x) \odot \varphi_2(x)) \, dx$. Scalar products of this type were systematically used in idempotent analysis. Using Theorems 2 and 7, one can easily describe a -linear functionals on idempotent spaces in terms of idempotent measures and integrals.

Example 10. Let X be a linear space in the traditional sense. The idempotent semiring (and linear space over $\mathbf{R}(\max, +)$) of convex functions $\text{Conv}(X, \mathbf{R})$ is b -complete but it is not a b -semialgebra over the semifield $K = \mathbf{R}(\max, +)$.

Any nonzero a -linear functional f on $\text{Conv}(X, \mathbf{R})$ has the form

$$\varphi \mapsto f(\varphi) = \sup_x \{\varphi(x) + \psi(x)\} = \int_X^{\oplus} \varphi(x) \odot \psi(x) dx,$$

where ψ is a concave function, i.e., an element of the idempotent space $\text{Conc}(X, \mathbf{R}) = -\text{Conv}(X, \mathbf{R})$.

7.4. Linear operator, b -semimodules and subsemimodules. In what follows, we suppose that all semigroups, semirings, semifields, semimodules, and spaces are idempotent unless otherwise specified. We fix a basic semiring K and examine semimodules and subsemimodules over K . We suppose that every linear functional takes its values in the basic semiring.

Let V and W be b -complete semimodules over a b -complete semiring K . Denote by $L_b(V, W)$ the set of all b -linear mappings from V to W . It is easy to check that $L_b(V, W)$ is an idempotent semigroup with respect to the pointwise addition of operators; the composition (product) of b -linear operators is also a b -linear operator, and therefore the set $L_b(V, V)$ is an idempotent semiring with respect to these operations, see, e.g., [49]. The following proposition can be treated as a version of the Banach–Steinhaus theorem in idempotent analysis (as well as Proposition 4 above).

Proposition 11. *Assume that S is a subset in $L_b(V, W)$ and the set $\{g(v) \mid g \in S\}$ is bounded in W for every element $v \in V$; thus the element $f(v) = \sup_{g \in S} g(v)$ exists, because the semimodule W is b -complete. Then the mapping $v \mapsto f(v)$ is a b -linear operator, i.e., an element of $L_b(V, W)$. The subset S is bounded; moreover, $\sup S = f$.*

Corollary 12. *The set $L_b(V, W)$ is a b -complete idempotent semigroup with respect to the (idempotent) pointwise addition of operators. If $V = W$, then $L_b(V, V)$ is a b -complete idempotent semiring with respect to the operations of pointwise addition and composition of operators.*

Corollary 13. *A subset S is bounded in $L_b(V, W)$ if and only if the set $\{g(v) \mid g \in S\}$ is bounded in the semimodule W for every element $v \in V$.*

A subset of an idempotent semimodule is called a *subsemimodule* if it is closed under addition and multiplication by scalar coefficients. A subsemimodule V of a b -complete semimodule W is *b -closed* if V is closed under sums of any subsets of V that are bounded in W . A subsemimodule of a b -complete semimodule is called a *b -subsemimodule* if the corresponding embedding is a b -homomorphism. It is easy to see that each b -closed subsemimodule is a b -subsemimodule, but the converse is not true. The main feature of b -subsemimodules is that restrictions of b -linear operators and functionals to these semimodules are b -linear.

The following definitions are very important for our purposes. Assume that W is an idempotent b -complete semimodule over a b -complete idempotent semiring K and V is a subset of W such that V is closed under multiplication by scalar coefficients and is an upper semilattice with respect to the order induced from W . Let us define an addition operation in V by the formula $x \oplus y = \sup\{x, y\}$, where \sup means the least upper bound in V . If K is a semifield, then V is a semimodule over K with respect to this addition.

For an arbitrary b -complete semiring K , we will say that V is a *quasisubsemimodule* of W if V is a semimodule with respect to this addition (this means that the corresponding distribution laws hold).

Recall that the symbol \wedge means the greatest lower bound (see Subsection 7.1 above). A quasisubsemimodule V of an idempotent b -complete semimodule W is called a *\wedge -subsemimodule* if it contains $\mathbf{0}$ and is closed under the operations of taking infima (greatest lower bounds) in W . It is easy to check that *each \wedge -subsemimodule is a b -complete semimodule*.

Note that quasisubsemimodules and \wedge -subsemimodules may fail to be subsemimodules, because only the order is induced and not the corresponding addition (see Example 18 below).

Recall that idempotent semimodules over semifields are *idempotent spaces*. In idempotent mathematics, such spaces are analogs of traditional linear

(vector) spaces over fields. In a similar way we use the corresponding terms like *b-spaces*, *b-subspaces*, *b-closed subspaces*, *\wedge -subspaces*, etc.

Some examples are presented below.

7.5. Functional semimodules. Let X be an arbitrary nonempty set and K be an idempotent semiring. By $K(X)$ denote the semimodule of all mappings (functions) $X \rightarrow K$ endowed with the pointwise operations. By $K_b(X)$ denote the subsemimodule of $K(X)$ consisting of all bounded mappings. If K is a *b*-complete semiring, then $K(X)$ and $K_b(X)$ are *b*-complete semimodules. Note that $K_b(X)$ is a *b*-subsemimodule but not a *b*-closed subsemimodule of $K(X)$. Given a point $x \in X$, by δ_x denote the functional on $K(X)$ that maps f to $f(x)$. It can easily be checked that the functional δ_x is *b*-linear on $K(X)$.

Recall that the functional δ_x is generated by the usual function

$$\delta_x(y) = \begin{cases} \mathbf{1}, & \text{if } x = y, \\ \mathbf{0}, & \text{if } x \neq y, \end{cases}$$

so $\varphi(x) = \int^\oplus \delta_x(y) \varphi(y) dy = \sup_y (\delta_x(y) \odot \varphi(y))$. Note that δ -functions form a natural (continuous in general) basis in any typical functional semimodule.

We say that a quasisubsemimodule of $K(X)$ is an (idempotent) *functional semimodule* on the set X . An idempotent functional semimodule in $K(X)$ is called *b-complete* if it is a *b*-complete semimodule.

A functional semimodule $V \subset K(X)$ is called a *functional b-semimodule* if it is a *b*-subsemimodule of $K(X)$; a functional semimodule $V \subset K(X)$ is called a *functional \wedge -semimodule* if it is a \wedge -subsemimodule of $K(X)$.

In general, a functional of the form δ_x on a functional semimodule is not even linear, much less *b*-linear (see Example 18 below). However, the following proposition holds, which is a direct consequence of our definitions.

Proposition 14. *An arbitrary b -complete functional semimodule W on a set X is a b -subsemimodule of $K(X)$ if and only if each functional of the form δ_x (where $x \in X$) is b -linear on W .*

Example 15. The semimodule $K_b(X)$ (consisting of all bounded mappings from an arbitrary set X to a b -complete idempotent semiring K) is a functional \wedge -semimodule. Hence it is a b -complete semimodule over K . Moreover, $K_b(X)$ is a b -subsemimodule of the semimodule $K(X)$ consisting of all mappings $X \rightarrow K$.

Example 16. If X is a finite set consisting of n elements ($n > 0$), then $K_b(X) = K(X)$ is an “ n -dimensional” semimodule over K ; it is denoted by K^n . In particular, \mathbf{R}_{max}^n is an idempotent space over the semifield \mathbf{R}_{max} , and $\widehat{\mathbf{R}}_{max}^n$ is a semimodule over the semiring $\widehat{\mathbf{R}}_{max}$. Note that $\widehat{\mathbf{R}}_{max}^n$ can be treated as a space over the semifield \mathbf{R}_{max} . For example, the semiring $\widehat{\mathbf{R}}_{max}$ can be treated as a space (semimodule) over \mathbf{R}_{max} .

Example 17. Let X be a topological space. Denote by $USC(X)$ the set of all upper semicontinuous functions with values in \mathbf{R}_{max} . By definition, a function $f(x)$ is upper semicontinuous if the set $X_s = \{x \in X \mid f(x) \geq s\}$ is closed in X for every element $s \in \mathbf{R}_{max}$ (see, e.g., [49], Sec. 2.8). If a family $\{f_\alpha\}$ consists of upper semicontinuous (e.g., continuous) functions and $f(x) = \inf_\alpha f_\alpha(x)$, then $f(x) \in USC(X)$. It is easy to check that $USC(X)$ has a natural structure of an idempotent space over \mathbf{R}_{max} . Moreover, $USC(X)$ is a functional \wedge -space on X and a b -space. The subspace $USC(X) \cap K_b(X)$ of $USC(X)$ consisting of bounded (from above) functions has the same properties.

Example 18. Note that an idempotent functional semimodule (and even a functional \wedge -semimodule) on a set X is not necessarily a subsemimodule of $K(X)$. The simplest example is the functional space (over $K = \mathbf{R}_{max}$) $\text{Conc}(\mathbf{R})$ consisting of all concave functions on \mathbf{R} with values in \mathbf{R}_{max} .

Recall that a function f belongs to $\text{Conc}(\mathbf{R})$ if and only if the subgraph of this function is convex, i.e., the formula $f(ax + (1 - a)y) \geq af(x) + (1 - a)f(y)$ is valid for $0 \leq a \leq 1$. The basic operations with $\mathbf{0} \in \mathbf{R}_{\max}$ can be defined in an obvious way. If $f, g \in \text{Conc}(\mathbf{R})$, then denote by $f \oplus g$ the sum of these functions in $\text{Conc}(\mathbf{R})$. The subgraph of $f \oplus g$ is the convex hull of the subgraphs of f and g . Thus $f \oplus g$ does not coincide with the pointwise sum (i.e., $\max\{f(x), g(x)\}$).

Example 19. Let X be a nonempty metric space with a fixed metric r . Denote by $\text{Lip}(X)$ the set of all functions defined on X with values in \mathbf{R}_{\max} satisfying the following *Lipschitz condition*:

$$|f(x) \odot (f(y))^{-1}| = |f(x) - f(y)| \leq r(x, y),$$

where x, y are arbitrary elements of X . The set $\text{Lip}(X)$ consists of continuous real-valued functions (but not all of them!) and (by definition) the function equal to $-\infty = \mathbf{0}$ at every point $x \in X$. The set $\text{Lip}(X)$ has the structure of an idempotent space over the semifield \mathbf{R}_{\max} . Spaces of the form $\text{Lip}(X)$ are said to be *Lipschitz spaces*. These spaces are b -subsemimodules in $K(X)$.

7.6. Integral representations of linear operators in functional semimodules. Let W be an idempotent b -complete semimodule over a b -complete semiring K and $V \subset K(X)$ be a b -complete functional semimodule on X . A mapping $A : V \rightarrow W$ is called an *integral operator* or an operator with an *integral representation* if there exists a mapping $k : X \rightarrow W$, called the *integral kernel* (or *kernel*) of the operator A , such that

$$(15) \quad Af = \sup_{x \in X} (f(x) \odot k(x)).$$

In idempotent analysis, the right-hand side of formula (11) is often written as $\int_X^\oplus f(x) \odot k(x) dx$. Regarding the kernel k , it is assumed that the set $\{f(x) \odot k(x) | x \in X\}$ is bounded in W for all $f \in V$ and $x \in X$. We denote the set of all functions with this property by $\text{kern}_{V,W}(X)$. In particular, if

$W = K$ and A is a functional, then this functional is called *integral*. Thus each integral functional can be presented in the form of a “scalar product” $f \mapsto \int_X^\oplus f(x) \odot k(x) dx$, where $k(x) \in K(X)$; in idempotent analysis, this situation is standard.

Note that a functional of the form δ_y (where $y \in X$) is a typical integral functional; in this case, $k(x) = \mathbf{1}$ if $x = y$ and $k(x) = \mathbf{0}$ otherwise.

We call a functional semimodule $V \subset K(X)$ *nondegenerate* if for every point $x \in X$ there exists a function $g \in V$ such that $g(x) = \mathbf{1}$, and *admissible* if for every function $f \in V$ and every point $x \in X$ such that $f(x) \neq \mathbf{0}$ there exists a function $g \in V$ such that $g(x) = \mathbf{1}$ and $f(x) \odot g \preceq f$.

Note that all idempotent functional semimodules over semifields are admissible (it is sufficient to set $g = f(x)^{-1} \odot f$).

Proposition 20. *Denote by X_V the subset of X defined by the formula $X_V = \{x \in X \mid \exists f \in V : f(x) = \mathbf{1}\}$. If the semimodule V is admissible, then the restriction to X_V defines an embedding $i : V \rightarrow K(X_V)$ and its image $i(V)$ is admissible and nondegenerate.*

If a mapping $k : X \rightarrow W$ is a kernel of a mapping $A : V \rightarrow W$, then the mapping $k_V : X \rightarrow W$ that is equal to k on X_V and equal to $\mathbf{0}$ on $X \setminus X_V$ is also a kernel of A .

A mapping $A : V \rightarrow W$ is integral if and only if the mapping $i_{-1}A : i(A) \rightarrow W$ is integral.

In what follows, K always denotes a fixed b -complete idempotent (basic) semiring. If an operator has an integral representation, this representation may not be unique. However, if the semimodule V is nondegenerate, then the set of all kernels of a fixed integral operator is bounded with respect to the natural order in the set of all kernels and is closed under the supremum operation applied to its arbitrary subsets. In particular, *any integral operator defined on a nondegenerate functional semimodule has a unique maximal kernel*.

An important point is that an integral operator is not necessarily b -linear and even linear except when V is a b -subsemimodule of $K(X)$ (see Proposition 21 below).

If W is a functional semimodule on a nonempty set Y , then an integral kernel k of an operator A can be naturally identified with the function on $X \times Y$ defined by the formula $k(x, y) = (k(x))(y)$. This function will also be called an *integral kernel* (or *kernel*) of the operator A . As a result, the set $\text{kern}_{V,W}(X)$ is identified with the set $\text{kern}_{V,W}(X, Y)$ of all mappings $k : X \times Y \rightarrow K$ such that for every point $x \in X$ the mapping $k_x : y \mapsto k(x, y)$ lies in W and for every $v \in V$ the set $\{v(x) \odot k_x | x \in X\}$ is bounded in W . Accordingly, the set of all integral kernels of b -linear operators can be embedded into $\text{kern}_{V,W}(X, Y)$.

If V and W are functional b -semimodules on X and Y , respectively, then the set of all kernels of b -linear operators can be identified with $\text{kern}_{V,W}(X, Y)$ and the following formula holds:

$$(16) \quad Af(y) = \sup_{x \in X} (f(x) \odot k(x, y)) = \int_X^{\oplus} f(x) \odot k(x, y) dx.$$

This formula coincides with the usual definition of an integral representation of an operator. Note that formula (15) can be rewritten in the form

$$(17) \quad Af = \sup_{x \in X} (\delta_x(f) \odot k(x)).$$

Proposition 21. *An arbitrary b -complete functional semimodule V on a nonempty set X is a functional b -semimodule on X (i.e., a b -subsemimodule of $K(X)$) if and only if all integral operators defined on V are b -linear.*

The following notion (definition) is especially important for our purposes. Let $V \subset K(X)$ be a b -complete functional semimodule over a b -complete idempotent semiring K . We say that the *kernel theorem* holds for the semimodule V if every b -linear mapping from V into an arbitrary b -complete semimodule over K has an integral representation.

Theorem 22. *Assume that a b -complete semimodule W over a b -complete semiring K and an admissible functional \wedge -semimodule $V \subset K(X)$ are given. Then every b -linear operator $A : V \rightarrow W$ has an integral representation of the form (15). In particular, if W is a functional b -semimodule on a set Y , then the operator A has an integral representation of the form (16). Thus for the semimodule V the kernel theorem holds.*

Remark 23. Examples of admissible functional \wedge -semimodules (and \wedge -spaces) appearing in Theorem 22 are presented above, see, e.g., examples 15 – 17. Thus for these functional semimodules and spaces V over K , the kernel theorem holds and every b -linear mapping V into an arbitrary b -complete semimodule W over K has an integral representation (16). Recall that every functional space over a b -complete semifield is admissible, see above.

7.7. Nuclear operators and their integral representations. Let us introduce some important definitions. Assume that V and W are b -complete semimodules. A mapping $g : V \rightarrow W$ is called *one-dimensional* (or a *mapping of rank 1*) if it is of the form $v \mapsto \phi(v) \odot w$, where ϕ is a b -linear functional on V and $w \in W$. A mapping g is called *b -nuclear* if it is the sum (i.e., supremum) of a bounded set of one-dimensional mappings. Since every one-dimensional mapping is b -linear (because the functional ϕ is b -linear), *every b -nuclear operator is b -linear* (see Corollary 12 above). Of course, b -nuclear mappings are closely related to tensor products of idempotent semimodules, see [48].

By $\phi \odot w$ we denote the one-dimensional operator $v \mapsto \phi(v) \odot w$. In fact, this is an element of the corresponding tensor product.

Proposition 24. *The composition (product) of a b -nuclear and a b -linear mapping or of a b -linear and a b -nuclear mapping is a b -nuclear operator.*

Theorem 25. *Assume that W is a b -complete semimodule over a b -complete semiring K and $V \subset K(X)$ is a functional b -semimodule. If every b -linear*

functional on V is integral, then a b -linear operator $A : V \rightarrow W$ has an integral representation if and only if it is b -nuclear.

7.8. The b -approximation property and b -nuclear semimodules and spaces. We say that a b -complete semimodule V has the *b -approximation property* if the identity operator $\text{id}: V \rightarrow V$ is b -nuclear (for a treatment of the approximation property for locally convex spaces in the traditional functional analysis, see [86]).

Let V be an arbitrary b -complete semimodule over a b -complete idempotent semiring K . We call this semimodule a *b -nuclear semimodule* if any b -linear mapping of V to an arbitrary b -complete semimodule W over K is a b -nuclear operator. Recall that, in the traditional functional analysis, a locally convex space is nuclear if and only if all continuous linear mappings of this space to any Banach space are nuclear operators, see [86].

Proposition 26. *Let V be an arbitrary b -complete semimodule over a b -complete semiring K . The following statements are equivalent:*

- 1 *the semimodule V has the b -approximation property;*
- 2 *every b -linear mapping from V to an arbitrary b -complete semimodule W over K is b -nuclear;*
- 3 *every b -linear mapping from an arbitrary b -complete semimodule W over K to the semimodule V is b -nuclear.*

Corollary 27. *An arbitrary b -complete semimodule over a b -complete semiring K is b -nuclear if and only if this semimodule has the b -approximation property.*

Recall that, in the traditional functional analysis, any nuclear space has the approximation property but the converse is not true.

Concrete examples of b -nuclear spaces and semimodules are described in Examples 15, 16 and 19 (see above). Important b -nuclear spaces and semimodules (e.g., the so-called Lipschitz spaces and semi-Lipschitz semimodules) are described in [57]. In this paper there is a description of

all functional b -semimodules for which the kernel theorem holds (as semi-Lipschitz semimodules); this result is due to G. B. Shpiz.

It is easy to show that the idempotent spaces $USC(X)$ and $\text{Conc}(\mathbf{R})$ (see Examples 17 and 18) are not b -nuclear (however, for these spaces the kernel theorem is true). The reason is that these spaces are not functional b -spaces and the corresponding δ -functionals are not b -linear (and even linear).

7.9. Kernel theorems for functional b -semimodules. Let $V \subset K(X)$ be a b -complete functional semimodule over a b -complete semiring K . Recall that for V the *kernel theorem* holds if every b -linear mapping of this semimodule to an arbitrary b -complete semimodule over K has an integral representation.

Theorem 28. *Assume that a b -complete semiring K and a nonempty set X are given. The kernel theorem holds for any functional b -semimodule $V \subset K(X)$ if and only if every b -linear functional on V is integral and the semimodule V is b -nuclear, i.e., has the b -approximation property.*

Corollary 29. *If for a functional b -semimodule the kernel theorem holds, then this semimodule is b -nuclear.*

Note that the possibility to obtain an integral representation of a functional means that one can decompose it into a sum of functionals of the form δ_x .

Corollary 30. *Assume that a b -complete semiring K and a nonempty set X are given. The kernel theorem holds for a functional b -semimodule $V \subset K(X)$ if and only if the identity operator $\text{id}: V \rightarrow V$ is integral.*

7.10. Integral representations of operators in abstract idempotent semimodules. In this subsection, we examine the following problem: when a b -complete idempotent semimodule V over a b -complete semiring is

isomorphic to a functional b -semimodule W such that the kernel theorem holds for W .

Assume that V is a b -complete idempotent semimodule over a b -complete semiring K and ϕ is a b -linear functional defined on V . We call this functional a δ -functional if there exists an element $v \in V$ such that

$$\phi(w) \odot v \preceq w$$

for every element $w \in V$. It is easy to see that every functional of the form δ_x is a δ -functional in this sense (but the converse is not true in general).

Denote by $\Delta(V)$ the set of all δ -functionals on V . Denote by i_Δ the natural mapping $V \rightarrow K(\Delta(V))$ defined by the formula

$$(i_\Delta(v))(\phi) = \phi(v)$$

for all $\phi \in \Delta(V)$. We say that an element $v \in V$ is *pointlike* if there exists a b -linear functional ϕ such that $\phi(w) \odot v \preceq w$ for all $w \in V$. The set of all pointlike elements of V will be denoted by $P(V)$. Recall that by $\phi \odot v$ we denote the one-dimensional operator $w \mapsto \phi(w) \odot v$.

The following assertion is an obvious consequence of our definitions (including the definition of the standard order) and the idempotency of our addition.

Remark 31. If a one-dimensional operator $\phi \odot v$ appears in the decomposition of the identity operator on V into a sum of one-dimensional operators, then $\phi \in \Delta(V)$ and $v \in P(V)$.

Denote by id and Id the identity operators on V and $i_\Delta(V)$, respectively.

Proposition 32. *If the operator id is b -nuclear, then i_Δ is an embedding and the operator Id is integral.*

If the operator i_Δ is an embedding and the operator Id is integral, then the operator id is b -nuclear.

Theorem 33. *A b -complete idempotent semimodule V over a b -complete idempotent semiring K is isomorphic to a functional b -semimodule for*

which the kernel theorem holds if and only if the identity mapping on V is a b -nuclear operator, i.e., V is a b -nuclear semimodule.

The following proposition shows that, in a certain sense, the embedding i_Δ is a universal representation of a b -nuclear semimodule in the form of a functional b -semimodule for which the kernel theorem holds.

Proposition 34. *Let K be a b -complete idempotent semiring, X be a nonempty set, and $V \subset K(X)$ be a functional b -semimodule on X for which the kernel theorem holds. Then there exists a natural mapping $i : X \rightarrow \Delta(V)$ such that the corresponding mapping $i_* : K(\Delta(V)) \rightarrow K(X)$ is an isomorphism of $i_\Delta(V)$ onto V .*

8. THE DEQUANTIZATION TRANSFORM, CONVEX GEOMETRY AND THE NEWTON POLYTOPES

Let X be a topological space. For functions $f(x)$ defined on X we shall say that a certain property is valid *almost everywhere* (a.e.) if it is valid for all elements x of an open dense subset of X . Suppose X is \mathbf{C}^n or \mathbf{R}^n ; denote by \mathbf{R}_+^n the set $x = \{(x_1, \dots, x_n) \in X \mid x_i \geq 0 \text{ for } i = 1, 2, \dots, n\}$. For $x = (x_1, \dots, x_n) \in X$ we set $\exp(x) = (\exp(x_1), \dots, \exp(x_n))$; so if $x \in \mathbf{R}^n$, then $\exp(x) \in \mathbf{R}_+^n$.

Denote by $\mathcal{F}(\mathbf{C}^n)$ the set of all functions defined and continuous on an open dense subset $U \subset \mathbf{C}^n$ such that $U \supset \mathbf{R}_+^n$. It is clear that $\mathcal{F}(\mathbf{C}^n)$ is a ring (and an algebra over \mathbf{C}) with respect to the usual addition and multiplications of functions.

For $f \in \mathcal{F}(\mathbf{C}^n)$ let us define the function \hat{f}_h by the following formula:

$$(18) \quad \hat{f}_h(x) = h \log |f(\exp(x/h))|,$$

where h is a (small) real positive parameter and $x \in \mathbf{R}^n$. Set

$$(19) \quad \hat{f}(x) = \lim_{h \rightarrow +0} \hat{f}_h(x),$$

if the right-hand side of (19) exists almost everywhere.

We shall say that the function $\hat{f}(x)$ is a *dequantization* of the function $f(x)$ and the map $f(x) \mapsto \hat{f}(x)$ is a *dequantization transform*. By construction, $\hat{f}_h(x)$ and $\hat{f}(x)$ can be treated as functions taking their values in \mathbf{R}_{\max} . Note that in fact $\hat{f}_h(x)$ and $\hat{f}(x)$ depend on the restriction of f to \mathbf{R}_+^n only; so in fact the dequantization transform is constructed for functions defined on \mathbf{R}_+^n only. It is clear that the dequantization transform is generated by the Maslov dequantization and the map $x \mapsto |x|$.

Of course, similar definitions can be given for functions defined on \mathbf{R}^n and \mathbf{R}_+^n . If $s = 1/h$, then we have the following version of (18) and (19):

$$(20) \quad \hat{f}(x) = \lim_{s \rightarrow \infty} (1/s) \log |f(e^{sx})|.$$

Denote by $\partial \hat{f}$ the subdifferential of the function \hat{f} at the origin.

If f is a polynomial we have

$$\partial \hat{f} = \{ v \in \mathbf{R}^n \mid (v, x) \leq \hat{f}(x) \ \forall x \in \mathbf{R}^n \}.$$

It is well known that all the convex compact subsets in \mathbf{R}^n form an idempotent semiring \mathcal{S} with respect to the Minkowski operations: for $\alpha, \beta \in \mathcal{S}$ the sum $\alpha \oplus \beta$ is the convex hull of the union $\alpha \cup \beta$; the product $\alpha \odot \beta$ is defined in the following way: $\alpha \odot \beta = \{ x \mid x = a + b, \text{ where } a \in \alpha, b \in \beta \}$, see Fig 3. In fact \mathcal{S} is an idempotent linear space over \mathbf{R}_{\max} .

Of course, the Newton polytopes of polynomials in n variables form a subsemiring \mathcal{N} in \mathcal{S} . If f, g are polynomials, then $\partial(\widehat{fg}) = \partial \hat{f} \odot \partial \hat{g}$; moreover, if f and g are “in general position”, then $\partial(\widehat{f+g}) = \partial \hat{f} \oplus \partial \hat{g}$. For the semiring of all polynomials with nonnegative coefficients the dequantization transform is a homomorphism of this “traditional” semiring to the idempotent semiring \mathcal{N} .

Theorem 35. *If f is a polynomial, then the subdifferential $\partial \hat{f}$ of \hat{f} at the origin coincides with the Newton polytope of f . For the semiring of polynomials with nonnegative coefficients, the transform $f \mapsto \partial \hat{f}$ is a homomorphism of this semiring to the semiring of convex polytopes with respect to the Minkowski operations (see above).*

Using the dequantization transform it is possible to generalize this result to a wide class of functions and convex sets, see below and [55].

8.1. Dequantization transform: algebraic properties. Denote by V the set \mathbf{R}^n treated as a linear Euclidean space (with the scalar product $(x, y) = x_1y_1 + x_2y_2 + \cdots + x_ny_n$) and set $V_+ = \mathbf{R}_+^n$. We shall say that a function $f \in \mathcal{F}(\mathbf{C}^n)$ is *dequantizable* whenever its dequantization $\hat{f}(x)$ exists (and is defined on an open dense subset of V). By $\mathcal{D}(\mathbf{C}^n)$ denote the set of all dequantizable functions and by $\widehat{\mathcal{D}}(V)$ denote the set $\{\hat{f} \mid f \in \mathcal{D}(\mathbf{C}^n)\}$. Recall that functions from $\mathcal{D}(\mathbf{C}^n)$ (and $\widehat{\mathcal{D}}(V)$) are defined almost everywhere and $f = g$ means that $f(x) = g(x)$ a.e., i.e., for x ranging over an open dense subset of \mathbf{C}^n (resp., of V). Denote by $\mathcal{D}_+(\mathbf{C}^n)$ the set of all functions $f \in \mathcal{D}(\mathbf{C}^n)$ such that $f(x_1, \dots, x_n) \geq 0$ if $x_i \geq 0$ for $i = 1, \dots, n$; so $f \in \mathcal{D}_+(\mathbf{C}^n)$ if the restriction of f to $V_+ = \mathbf{R}_+^n$ is a nonnegative function. By $\widehat{\mathcal{D}}_+(V)$ denote the image of $\mathcal{D}_+(\mathbf{C}^n)$ under the dequantization transform. We shall say that functions $f, g \in \mathcal{D}(\mathbf{C}^n)$ are in *general position* whenever $\hat{f}(x) \neq \hat{g}(x)$ for x running an open dense subset of V .

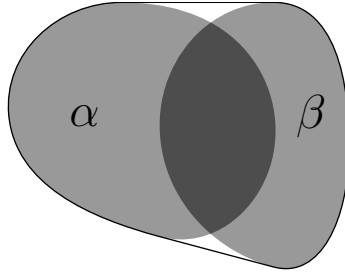


FIGURE 3. Algebra of convex subsets.

Theorem 36. *For functions $f, g \in \mathcal{D}(\mathbf{C}^n)$ and any nonzero constant c , the following equations are valid:*

- 1) $\widehat{fg} = \widehat{f} + \widehat{g}$;
- 2) $|\widehat{f}| = \widehat{f}$; $\widehat{cf} = f$; $\widehat{c} = 0$;
- 3) $\widehat{(f+g)}(x) = \max\{\widehat{f}(x), \widehat{g}(x)\}$ a.e. if f and g are nonnegative on V_+ (i.e., $f, g \in \mathcal{D}_+(\mathbf{C}^n)$) or f and g are in general position.

Left-hand sides of these equations are well-defined automatically.

Corollary 37. *The set $\mathcal{D}_+(\mathbf{C}^n)$ has a natural structure of a semiring with respect to the usual addition and multiplication of functions taking their values in \mathbf{C} . The set $\widehat{\mathcal{D}}_+(V)$ has a natural structure of an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$; elements of $\widehat{\mathcal{D}}_+(V)$ can be naturally treated as functions taking their values in \mathbf{R}_{\max} . The dequantization transform generates a homomorphism from $\mathcal{D}_+(\mathbf{C}^n)$ to $\widehat{\mathcal{D}}_+(V)$.*

8.2. Generalized polynomials and simple functions. For any nonzero number $a \in \mathbf{C}$ and any vector $d = (d_1, \dots, d_n) \in V = \mathbf{R}^n$ we set $m_{a,d}(x) = a \prod_{i=1}^n x_i^{d_i}$; functions of this kind we shall call *generalized monomials*. Generalized monomials are defined a.e. on \mathbf{C}^n and on V_+ , but not on V unless the numbers d_i take integer or suitable rational values. We shall say that a function f is a *generalized polynomial* whenever it is a finite sum of linearly independent generalized monomials. For instance, Laurent polynomials and Puiseux polynomials are examples of generalized polynomials.

As usual, for $x, y \in V$ we set $(x, y) = x_1 y_1 + \dots + x_n y_n$. The following proposition is a result of a trivial calculation.

Proposition 38. *For any nonzero number $a \in V = \mathbf{C}$ and any vector $d \in V = \mathbf{R}^n$ we have $(\widehat{m_{a,d}})_h(x) = (d, x) + h \log |a|$.*

Corollary 39. *If f is a generalized monomial, then \widehat{f} is a linear function.*

Recall that a real function p defined on $V = \mathbf{R}^n$ is *sublinear* if $p = \sup_{\alpha} p_{\alpha}$, where $\{p_{\alpha}\}$ is a collection of linear functions. Sublinear functions defined everywhere on $V = \mathbf{R}^n$ are convex; thus these functions are continuous, see [61]. We discuss sublinear functions of this kind only. Suppose p is a continuous function defined on V , then p is sublinear whenever

- 1) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$;
- 2) $p(cx) = cp(x)$ for all $x \in V$, $c \in \mathbf{R}_+$.

So if p_1, p_2 are sublinear functions, then $p_1 + p_2$ is a sublinear function.

We shall say that a function $f \in \mathcal{F}(\mathbf{C}^n)$ is *simple*, if its dequantization \hat{f} exists and a.e. coincides with a sublinear function; by misuse of language, we shall denote this (uniquely defined everywhere on V) sublinear function by the same symbol \hat{f} .

Recall that simple functions f and g are *in general position* if $\hat{f}(x) \neq \hat{g}(x)$ for all x belonging to an open dense subset of V . In particular, generalized monomials are in general position whenever they are linearly independent.

Denote by $\text{Sim}(\mathbf{C}^n)$ the set of all simple functions defined on V and denote by $\text{Sim}_+(\mathbf{C}^n)$ the set $\text{Sim}(\mathbf{C}^n) \cap \mathcal{D}_+(\mathbf{C}^n)$. By $\text{Sbl}(V)$ denote the set of all (continuous) sublinear functions defined on $V = \mathbf{R}^n$ and by $\text{Sbl}_+(V)$ denote the image $\widehat{\text{Sim}_+(\mathbf{C}^n)}$ of $\text{Sim}_+(\mathbf{C}^n)$ under the dequantization transform.

The following statements can be easily deduced from Theorem 8.2 and definitions.

Corollary 40. *The set $\text{Sim}_+(\mathbf{C}^n)$ is a subsemiring of $\mathcal{D}_+(\mathbf{C}^n)$ and $\text{Sbl}_+(V)$ is an idempotent subsemiring of $\widehat{\mathcal{D}_+}(V)$. The dequantization transform generates an epimorphism of $\text{Sim}_+(\mathbf{C}^n)$ onto $\text{Sbl}_+(V)$. The set $\text{Sbl}(V)$ is an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$.*

Corollary 41. *Polynomials and generalized polynomials are simple functions.*

We shall say that functions $f, g \in \mathcal{D}(V)$ are *asymptotically equivalent* whenever $\hat{f} = \hat{g}$; any simple function f is an *asymptotic monomial* whenever \hat{f} is a linear function. A simple function f will be called an *asymptotic polynomial* whenever \hat{f} is a sum of a finite collection of nonequivalent asymptotic monomials.

Corollary 42. *Every asymptotic polynomial is a simple function.*

Example 43. Generalized polynomials, logarithmic functions of (generalized) polynomials, and products of polynomials and logarithmic functions are asymptotic polynomials. This follows from our definitions and formula (19).

8.3. Subdifferentials of sublinear functions. We shall use some elementary results from convex analysis. These results can be found, e.g., in [61], ch. 1, §1.

For any function $p \in Sbl(V)$ we set

$$(21) \quad \partial p = \{ v \in V \mid (v, x) \leq p(x) \ \forall x \in V \}.$$

It is well known from convex analysis that for any sublinear function p the set ∂p is exactly the *subdifferential* of p at the origin. The following propositions are also known in convex analysis.

Proposition 44. *Suppose $p_1, p_2 \in Sbl(V)$, then*

- 1) $\partial(p_1 + p_2) = \partial p_1 \odot \partial p_2 = \{ v \in V \mid v = v_1 + v_2, \text{ where } v_1 \in \partial p_1, v_2 \in \partial p_2 \};$
- 2) $\partial(\max\{p_1(x), p_2(x)\}) = \partial p_1 \oplus \partial p_2.$

Recall that $\partial p_1 \oplus \partial p_2$ is a convex hull of the set $\partial p_1 \cup \partial p_2$.

Proposition 45. *Suppose $p \in Sbl(V)$. Then ∂p is a nonempty convex compact subset of V .*

Corollary 46. *The map $p \mapsto \partial p$ is a homomorphism of the idempotent semiring $Sbl(V)$ (see Corollary 37) to the idempotent semiring \mathcal{S} of all convex compact subsets of V (see Subsection 8.1 above).*

8.4. Newton sets for simple functions. For any simple function $f \in \text{Sim}(\mathbf{C}^n)$ let us denote by $N(f)$ the set $\partial(\hat{f})$. We shall call $N(f)$ the *Newton set* of the function f .

Proposition 47. *For any simple function f , its Newton set $N(f)$ is a nonempty convex compact subset of V .*

This proposition follows from Proposition 45 and definitions.

Theorem 48. *Suppose that f and g are simple functions. Then*

- 1) $N(fg) = N(f) \odot N(g) = \{v \in V \mid v = v_1 + v_2 \text{ with } v_1 \in N(f), v_2 \in N(g)\}$;
- 2) $N(f + g) = N(f) \oplus N(g)$, if f_1 and f_2 are in general position or $f_1, f_2 \in \text{Sim}_+(\mathbf{C}^n)$ (recall that $N(f) \oplus N(g)$ is the convex hull of $N(f) \cup N(g)$).

This theorem follows from Theorem 36, Proposition 44 and definitions.

Corollary 49. *The map $f \mapsto N(f)$ generates a homomorphism from $\text{Sim}_+(\mathbf{C}^n)$ to \mathcal{S} .*

Proposition 50. *Let $f = m_{a,d}(x) = a \prod_{i=1}^n x_i^{d_i}$ be a monomial; here $d = (d_1, \dots, d_n) \in V = \mathbf{R}^n$ and a is a nonzero complex number. Then $N(f) = \{d\}$.*

This follows from Proposition 38, Corollary 39 and definitions.

Corollary 51. *Let $f = \sum_{d \in D} m_{a_d, d}$ be a polynomial. Then $N(f)$ is the polytope $\oplus_{d \in D} \{d\}$, i.e. the convex hull of the finite set D .*

This statement follows from Theorem 48 and Proposition 50. Thus in this case $N(f)$ is the well-known classical Newton polytope of the polynomial f .

Now the following corollary is obvious.

Corollary 52. *Let f be a generalized or asymptotic polynomial. Then its Newton set $N(f)$ is a convex polytope.*

Example 53. . Consider the one dimensional case, i.e., $V = \mathbf{R}$ and suppose $f_1 = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ and $f_2 = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$, where $a_n \neq 0$, $b_m \neq 0$, $a_0 \neq 0$, $b_0 \neq 0$. Then $N(f_1)$ is the segment $[0, n]$ and $N(f_2)$ is the segment $[0, m]$. So the map $f \mapsto N(f)$ corresponds to the map $f \mapsto \deg(f)$, where $\deg(f)$ is a degree of the polynomial f . In this case Theorem 2 means that $\deg(fg) = \deg f + \deg g$ and $\deg(f + g) = \max\{\deg f, \deg g\} = \max\{n, m\}$ if $a_i \geq 0$, $b_i \geq 0$ or f and g are in general position.

9. DEQUANTIZATION OF SET FUNCTIONS AND MEASURES ON METRIC SPACES

The following results are presented in [56].

Example 54. Let M be a metric space, S its arbitrary subset with a compact closure. It is well-known that a Euclidean d -dimensional ball B_ρ of radius ρ has volume

$$\text{vol}_d(B_\rho) = \frac{\Gamma(1/2)^d}{\Gamma(1 + d/2)} \rho^d,$$

where d is a natural parameter. By means of this formula it is possible to define a volume of B_ρ for any *real* d . Cover S by a finite number of balls of radii ρ_m . Set

$$v_d(S) := \lim_{\rho \rightarrow 0} \inf_{\rho_m < \rho} \sum_m \text{vol}_d(B_{\rho_m}).$$

Then there exists a number D such that $v_d(S) = 0$ for $d > D$ and $v_d(S) = \infty$ for $d < D$. This number D is called the *Hausdorff-Besicovich dimension* (or *HB-dimension*) of S , see, e.g., [67]. Note that a set of non-integral HB-dimension is called a fractal in the sense of B. Mandelbrot.

Theorem 55. Denote by $\mathcal{N}_\rho(S)$ the minimal number of balls of radius ρ covering S . Then

$$D(S) = \varliminf_{\rho \rightarrow +0} \log_\rho(\mathcal{N}_\rho(S)^{-1}),$$

where $D(S)$ is the HB-dimension of S . Set $\rho = e^{-s}$, then

$$D(S) = \varliminf_{s \rightarrow +\infty} (1/s) \cdot \log \mathcal{N}_{\exp(-s)}(S).$$

So the HB-dimension $D(S)$ can be treated as a result of a dequantization of the set function $\mathcal{N}_\rho(S)$.

Example 56. Let μ be a set function on M (e.g., a probability measure) and suppose that $\mu(B_\rho) < \infty$ for every ball B_ρ . Let $B_{x,\rho}$ be a ball of radius ρ having the point $x \in M$ as its center. Then define $\mu_x(\rho) := \mu(B_{x,\rho})$ and let $\rho = e^{-s}$ and

$$D_{x,\mu} := \varliminf_{s \rightarrow +\infty} -(1/s) \cdot \log(|\mu_x(e^{-s})|).$$

This number could be treated as a dimension of M at the point x with respect to the set function μ . So this dimension is a result of a dequantization of the function $\mu_x(\rho)$, where x is fixed. There are many dequantization procedures of this type in different mathematical areas. In particular, V.P. Maslov's negative dimension (see [67]) can be treated similarly.

10. DEQUANTIZATION OF GEOMETRY

An idempotent version of real algebraic geometry was discovered in the report of O. Viro for the Barcelona Congress [94]. Starting from the idempotent correspondence principle O. Viro constructed a piecewise-linear geometry of polyhedra of a special kind in finite dimensional Euclidean spaces as a result of the Maslov dequantization of real algebraic geometry. He indicated important applications in real algebraic geometry (e.g., in the framework of Hilbert's 16th problem for constructing real algebraic varieties with prescribed properties and parameters) and relations to complex algebraic geometry and amoebas in the sense of I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, see [25, 95]. Then complex algebraic geometry was dequantized by G. Mikhalkin and the result turned out to be the same; this

new ‘idempotent’ (or asymptotic) geometry is now often called the *tropical algebraic geometry*, see, e.g., [32, 43, 46, 53, 72, 73].

There is a natural relation between the Maslov dequantization and amoebas.

Suppose $(\mathbf{C}^*)^n$ is a complex torus, where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ is the group of nonzero complex numbers under multiplication. For $z = (z_1, \dots, z_n) \in (\mathbf{C}^*)^n$ and a positive real number h denote by $\text{Log}_h(z) = h \log(|z|)$ the element

$$(h \log |z_1|, h \log |z_2|, \dots, h \log |z_n|) \in \mathbf{R}^n.$$

Suppose $V \subset (\mathbf{C}^*)^n$ is a complex algebraic variety; denote by $\mathcal{A}_h(V)$ the set $\text{Log}_h(V)$. If $h = 1$, then the set $\mathcal{A}(V) = \mathcal{A}_1(V)$ is called the *amoeba* of V ; the amoeba $\mathcal{A}(V)$ is a closed subset of \mathbf{R}^n with a non-empty complement. Note that this construction depends on our coordinate system.

For the sake of simplicity suppose V is a hypersurface in $(\mathbf{C}^*)^n$ defined by a polynomial f ; then there is a deformation $h \mapsto f_h$ of this polynomial generated by the Maslov dequantization and $f_h = f$ for $h = 1$. Let $V_h \subset (\mathbf{C}^*)^n$ be the zero set of f_h and set $\mathcal{A}_h(V_h) = \text{Log}_h(V_h)$. Then there exists a tropical variety $\text{Tro}(V)$ such that the subsets $\mathcal{A}_h(V_h) \subset \mathbf{R}^n$ tend to $\text{Tro}(V)$ in the Hausdorff metric as $h \rightarrow 0$. The tropical variety $\text{Tro}(V)$ is a result of a deformation of the amoeba $\mathcal{A}(V)$ and the Maslov dequantization of the variety V . The set $\text{Tro}(V)$ is called the *skeleton* of $\mathcal{A}(V)$.

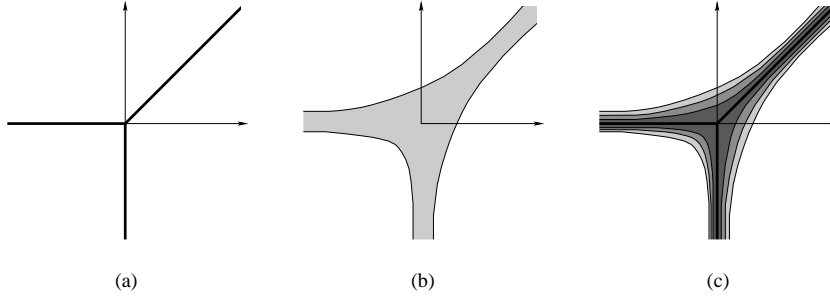


FIGURE 4. Tropical line and deformations of an amoeba

Example 57. For the line $V = \{ (x, y) \in (\mathbf{C}^*)^2 \mid x+y+1 = 0 \}$ the piecewise-linear graph $Tro(V)$ is a tropical line, see Fig. 4(a). The amoeba $\mathcal{A}(V)$ is represented in Fig. 4(b), while Fig. 4(c) demonstrates the corresponding deformation of the amoeba.

11. SOME SEMIRING CONSTRUCTIONS AND THE MATRIX BELLMAN EQUATION

11.1. Complete idempotent semirings and examples. Recall that a partially ordered set S is *complete* if for every subset $T \subset S$ there exist elements $\sup T \in S$ and $\inf T \in S$. We say that an idempotent semiring S is *complete* if it is complete as an ordered set with respect to the standard order. Of course, any a-complete semiring (see subsect. 7.1) is complete. The most well-known and important examples are “numerical semirings” consisting of (a subset of) real numbers and ordered by the usual linear order \leq .

Example 58. Consider the semiring $\widehat{\mathbf{R}}_{\max} = \mathbf{R}_{\max} \cup \{\infty\}$ with standard operations $\oplus = \max$, $\odot = +$ and neutral elements $\mathbf{0} = -\infty$, $\mathbf{1} = 0$, $x \leq \infty$, $x \oplus \infty = \infty$ for all x , $x \odot \infty = \infty \odot x = \infty$ if $x \neq \mathbf{0}$, and $\mathbf{0} \odot \infty = \infty \odot \mathbf{0}$. The semiring $\widehat{\mathbf{R}}_{\max}$ is complete and a-complete. The semiring $\widehat{\mathbf{R}}_{\min} = \mathbf{R}_{\min} \cup \{-\infty\}$ with obvious operations is also complete; $\widehat{\mathbf{R}}_{\min}$ and $\widehat{\mathbf{R}}_{\max}$ are isomorphic.

Example 59. Consider the semiring $S_{\max, \min}^{[a, b]}$ defined on the real interval $[a, b]$ with operations $\oplus = \max$, $\odot = \min$ and neutral elements $\mathbf{0} = a$ and $\mathbf{1} = b$. The semiring is complete and a-complete. Set $S_{\max, \min} = S_{\max, \min}^{[a, b]}$ with $a = -\infty$ and $b = +\infty$. If $-\infty \leq a < b \leq +\infty$ then $S_{\max, \min}^{[a, b]}$ and $S_{\max, \min}$ are isomorphic.

Example 60. The Boolean algebra $B = \{\mathbf{0}, \mathbf{1}\}$ is a complete and a-complete semifield consisting of two elements.

11.2. Closure operations. Let a semiring S be endowed with a partial unary *closure (or Kleene) operation* $*$ such that $x \preceq y$ implies $x^* \preceq y^*$ and $x^* = \mathbf{1} \oplus (x^* \odot x) = \mathbf{1} \oplus (x \odot x^*)$ on its domain of definition. In particular, $\mathbf{0}^* = \mathbf{1}$ by definition. These axioms imply that $x^* = \mathbf{1} \oplus x \oplus x^2 \oplus \cdots \oplus (x^* \odot x^n)$ if $n \geq 1$. Thus x^* can be considered as a ‘regularized sum’ of the series $x^* = \mathbf{1} \oplus x \oplus x^2 \oplus \cdots$; in an idempotent semiring, by definition, $x^* = \sup\{\mathbf{1}, x, x^2, \dots\}$ if this supremum exists. So if S is complete, then the closure operation is well-defined for every element $x \in S$.

In numerical semirings the operation $*$ is defined as follows: $x^* = (1 - x)^{-1}$ if $x \prec 1$ in \mathbf{R}_+ , or $\widehat{\mathbf{R}}_+$ and $x^* = \infty$ if $x \succcurlyeq 1$ in $\widehat{\mathbf{R}}_+$; $x^* = \mathbf{1}$ if $x \preceq \mathbf{1}$ in \mathbf{R}_{\max} and $\widehat{\mathbf{R}}_{\max}$, $x^* = \infty$ if $x \succ \mathbf{1}$ in $\widehat{\mathbf{R}}_{\max}$, $x^* = \mathbf{1}$ for all x in $S_{\max, \min}^{[a, b]}$. In all other cases x^* is undefined. Note that the closure operation is very easy to implement.

11.3. Matrices over semirings. Denote by $\text{Mat}_{mn}(S)$ a set of all matrices $A = (a_{ij})$ with m rows and n columns whose coefficients belong to a semiring S . The sum $A \oplus B$ of matrices $A, B \in \text{Mat}_{mn}(S)$ and the product AB of matrices $A \in \text{Mat}_{lm}(S)$ and $B \in \text{Mat}_{mn}(S)$ are defined according to the usual rules of linear algebra: $A \oplus B = (a_{ij} \oplus b_{ij}) \in \text{Mat}_{mn}(S)$ and

$$AB = \left(\bigoplus_{k=1}^m a_{ik} \odot b_{kj} \right) \in \text{Mat}_{ln}(S),$$

where $A \in \text{Mat}_{lm}(S)$ and $B \in \text{Mat}_{mn}(S)$. Note that we write AB instead of $A \odot B$.

If the semiring S is ordered, then the set $\text{Mat}_{mn}(S)$ is ordered by the relation $A = (a_{ij}) \preceq B = (b_{ij})$ iff $a_{ij} \preceq b_{ij}$ in S for all $1 \leq i \leq m$, $1 \leq j \leq n$.

The matrix multiplication is consistent with the order \preceq in the following sense: if $A, A' \in \text{Mat}_{lm}(S)$, $B, B' \in \text{Mat}_{mn}(S)$ and $A \preceq A'$, $B \preceq B'$, then $AB \preceq A'B'$ in $\text{Mat}_{ln}(S)$. The set $\text{Mat}_{nn}(S)$ of square $(n \times n)$ matrices over an idempotent semiring S forms a idempotent semiring with a zero element

$O = (o_{ij})$, where $o_{ij} = \mathbf{0}$, $1 \leq i, j \leq n$, and a unit element $I = (\delta_{ij})$, where $\delta_{ij} = \mathbf{1}$ if $i = j$ and $\delta_{ij} = \mathbf{0}$ otherwise.

The set Mat_{nn} is an example of a noncommutative semiring if $n > 1$.

The closure operation in matrix semirings over an idempotent semiring S can be defined inductively (another way to do that see in [26] and below): $A^* = (a_{11})^* = (a_{11}^*)$ in $\text{Mat}_{11}(S)$ and for any integer $n > 1$ and any matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in \text{Mat}_{kk}(S)$, $A_{12} \in \text{Mat}_{k, n-k}(S)$, $A_{21} \in \text{Mat}_{n-k, k}(S)$, $A_{22} \in \text{Mat}_{n-k, n-k}(S)$, $1 \leq k \leq n$, by definition,

$$(22) \quad A^* = \begin{pmatrix} A_{11}^* \oplus A_{11}^* A_{12} D^* A_{21} A_{11}^* & A_{11}^* A_{12} D^* \\ D^* A_{21} A_{11}^* & D^* \end{pmatrix},$$

where $D = A_{22} \oplus A_{21} A_{11}^* A_{12}$. It can be proved that this definition of A^* implies that the equality $A^* = A^* A \oplus I$ is satisfied and thus A^* is a ‘regularized sum’ of the series $I \oplus A \oplus A^2 \oplus \dots$.

Note that this recurrence relation coincides with the formulas of escalator method of matrix inversion in the traditional linear algebra over the field of real or complex numbers, up to the algebraic operations used. Hence this algorithm of matrix closure requires a polynomial number of operations in n .

11.4. Discrete stationary Bellman equations. Let S be a semiring. The *discrete stationary Bellman equation* has the form

$$(23) \quad X = AX \oplus B,$$

where $A \in \text{Mat}_{nn}(S)$, $X, B \in \text{Mat}_{ns}(S)$, and the matrix X is unknown. Let A^* be the closure of the matrix A . It follows from the identity $A^* = A^* A \oplus I$ that the matrix $A^* B$ satisfies this equation; moreover, it can be proved that for idempotent semirings this solution is the least in the set of solutions to equation (23) with respect to the partial order in $\text{Mat}_{ns}(S)$.

Equation (23) over max-plus semiring arises in connection with Bellman optimality principle and discretization of Hamilton-Jacobi equations, see e.g., [71]. It is also intimately related with optimization problems on graphs to be discussed below.

11.5. Weighted directed graphs and matrices over semirings. Suppose that S is a semiring with zero $\mathbf{0}$ and unity $\mathbf{1}$. It is well-known that any square matrix $A = (a_{ij}) \in \text{Mat}_{nn}(S)$ specifies a *weighted directed graph*. This geometrical construction includes three kinds of objects: the set X of n elements x_1, \dots, x_n called *nodes*, the set Γ of all ordered pairs (x_i, x_j) such that $a_{ij} \neq \mathbf{0}$ called *arcs*, and the mapping $A: \Gamma \rightarrow S$ such that $A(x_i, x_j) = a_{ij}$. The elements a_{ij} of the semiring S are called *weights* of the arcs. See Fig. 5

Conversely, any given weighted directed graph with n nodes specifies a unique matrix $A \in \text{Mat}_{nn}(S)$.

This definition allows for some pairs of nodes to be disconnected if the corresponding element of the matrix A is $\mathbf{0}$ and for some channels to be “loops” with coincident ends if the matrix A has nonzero diagonal elements.

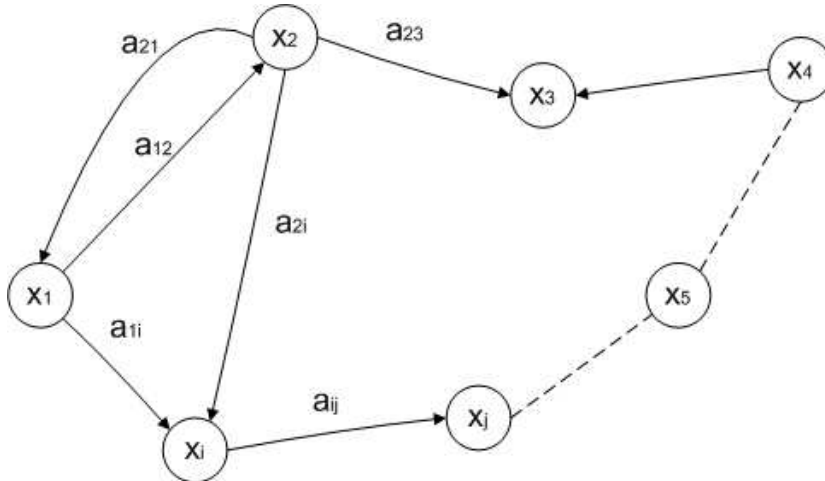


FIGURE 5. A weighted directed graph.

This concept is convenient for analysis of parallel and distributed computations and design of computing media and networks (see, e.g., [5, 45, 69, 97]).

Recall that a sequence of nodes of the form

$$p = (y_0, y_1, \dots, y_k)$$

with $k \geq 0$ and $(y_i, y_{i+1}) \in \Gamma$, $i = 0, \dots, k-1$, is called a *path* of length k connecting y_0 with y_k . Denote the set of all such paths by $P_k(y_0, y_k)$. The weight $A(p)$ of a path $p \in P_k(y_0, y_k)$ is defined to be the product of weights of arcs connecting consecutive nodes of the path:

$$A(p) = A(y_0, y_1) \odot \dots \odot A(y_{k-1}, y_k).$$

By definition, for a ‘path’ $p \in P_0(x_i, x_j)$ of length $k = 0$ the weight is $\mathbf{1}$ if $i = j$ and $\mathbf{0}$ otherwise.

For each matrix $A \in \text{Mat}_{nn}(S)$ define $A^0 = I = (\delta_{ij})$ (where $\delta_{ij} = \mathbf{1}$ if $i = j$ and $\delta_{ij} = \mathbf{0}$ otherwise) and $A^k = AA^{k-1}$, $k \geq 1$. Let $a_{ij}^{(k)}$ be the (i, j) th element of the matrix A^k . It is easily checked that

$$a_{ij}^{(k)} = \bigoplus_{\substack{i_0=i, i_k=j \\ 1 \leq i_1, \dots, i_{k-1} \leq n}} a_{i_0 i_1} \odot \dots \odot a_{i_{k-1} i_k}.$$

Thus $a_{ij}^{(k)}$ is the supremum of the set of weights corresponding to all paths of length k connecting the node $x_{i_0} = x_i$ with $x_{i_k} = x_j$.

Denote the elements of the matrix A^* by $a_{ij}^{(*)}$, $i, j = 1, \dots, n$; then

$$a_{ij}^{(*)} = \bigoplus_{0 \leq k < \infty} \bigoplus_{p \in P_k(x_i, x_j)} A(p).$$

The closure matrix A^* solves the well-known *algebraic path problem*, which is formulated as follows: for each pair (x_i, x_j) calculate the supremum of weights of all paths (of arbitrary length) connecting node x_i with node x_j . The closure operation in matrix semirings has been studied extensively (see, e.g., [1, 2, 6–8, 14, 15, 20–22, 26–30, 33, 34, 59] and references therein).

Example 61 (The shortest path problem.). Let $S = \mathbf{R}_{\min}$, so the weights are real numbers. In this case

$$A(p) = A(y_0, y_1) + A(y_1, y_2) + \cdots + A(y_{k-1}, y_k).$$

If the element a_{ij} specifies the length of the arc (x_i, x_j) in some metric, then $a_{ij}^{(*)}$ is the length of the shortest path connecting x_i with x_j .

Example 62 (The maximal path width problem.). Let $S = \mathbf{R} \cup \{0, 1\}$ with $\oplus = \max$, $\odot = \min$. Then

$$a_{ij}^{(*)} = \max_{p \in \bigcup_{k \geq 1} P_k(x_i, x_j)} A(p), \quad A(p) = \min(A(y_0, y_1), \dots, A(y_{k-1}, y_k)).$$

If the element a_{ij} specifies the “width” of the arc (x_i, x_j) , then the width of a path p is defined as the minimal width of its constituting arcs and the element $a_{ij}^{(*)}$ gives the supremum of possible widths of all paths connecting x_i with x_j .

Example 63 (A simple dynamic programming problem.). Let $S = \mathbf{R}_{\max}$ and suppose a_{ij} gives the *profit* corresponding to the transition from x_i to x_j . Define the vector $B = (b_i) \in \text{Mat}_{n1}(\mathbf{R}_{\max})$ whose element b_i gives the *terminal profit* corresponding to exiting from the graph through the node x_i . Of course, negative profits (or, rather, losses) are allowed. Let m be the total profit corresponding to a path $p \in P_k(x_i, x_j)$, i.e.

$$m = A(p) + b_j.$$

Then it is easy to check that the supremum of profits that can be achieved on paths of length k beginning at the node x_i is equal to $(A^k B)_i$ and the supremum of profits achievable without a restriction on the length of a path equals $(A^* B)_i$.

Example 64 (The matrix inversion problem.). Note that in the formulas of this section we are using distributivity of the multiplication \odot with respect to the addition \oplus but do not use the idempotency axiom. Thus

the algebraic path problem can be posed for a nonidempotent semiring S as well (see, e.g., [84]). For instance, if $S = \mathbf{R}$, then

$$A^* = I + A + A^2 + \cdots = (I - A)^{-1}.$$

If $\|A\| > 1$ but the matrix $I - A$ is invertible, then this expression defines a regularized sum of the divergent matrix power series $\sum_{i \geq 0} A^i$.

There are many other important examples of problems (in different areas) related to algorithms of linear algebra over semirings (transitive closures of relations, accessible sets, critical paths, paths of greatest capacities, the most reliable paths, interval and other problems), see [1, 2, 5–7, 12, 14–17, 20–24, 26–31, 33, 34, 58, 59, 69, 75, 76, 81–84, 87, 89, 98–101].

We emphasize that this connection between the matrix closure operation and solution to the Bellman equation gives rise to a number of different algorithms for numerical calculation of the closure matrix. All these algorithms are adaptations of the well-known algorithms of the traditional computational linear algebra, such as the Gauss–Jordan elimination, various iterative and escalator schemes, etc. This is a special case of the idempotent superposition principle.

In fact, the theory of the discrete stationary Bellman equation can be developed using the identity $A^* = AA^* \oplus I$ as an additional axiom without any substantial interpretation (the so-called *closed semirings*, see, e.g., [7, 26, 38, 84]).

12. UNIVERSAL ALGORITHMS

Computational algorithms are constructed on the basis of certain primitive operations. These operations manipulate data that describe “numbers.” These “numbers” are elements of a “numerical domain,” i.e., a mathematical object such as the field of real numbers, the ring of integers, or an idempotent semiring of numbers.

In practice elements of the numerical domains are replaced by their computer representations, i.e., by elements of certain finite models of these domains. Examples of models that can be conveniently used for computer representation of real numbers are provided by various modifications of floating point arithmetics, approximate arithmetics of rational numbers [52], and interval arithmetics. The difference between mathematical objects (“ideal” numbers) and their finite models (computer representations) results in computational (e.g., rounding) errors.

An algorithm is called *universal* if it is independent of a particular numerical domain and/or its computer representation. A typical example of a universal algorithm is the computation of the scalar product (x, y) of two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ by the formula $(x, y) = x_1y_1 + \dots + x_ny_n$. This algorithm (formula) is independent of a particular domain and its computer implementation, since the formula is well-defined for any semiring. It is clear that one algorithm can be more universal than another. For example, the simplest Newton–Cotes formula, the rectangular rule, provides the most universal algorithm for numerical integration; indeed, this formula is valid even for idempotent integration (over any idempotent semiring, see above and [5, 33, 39, 40, 42–44, 51, 62–65]). Other quadrature formulas (e.g., combined trapezoid rule or the Simpson formula) are independent of computer arithmetics and can be used (e.g., in an iterative form) for computations with arbitrary accuracy. In contrast, algorithms based on Gauss–Jacobi formulas are designed for fixed accuracy computations: they include constants (coefficients and nodes of these formulas) defined with fixed accuracy. Certainly, algorithms of this type can be made more universal by including procedures for computing the constants; however, this results in an unjustified complication of the algorithms.

Computer algebra algorithms used in such systems as Mathematica, Maple, REDUCE, and others are highly universal. Most of the standard

algorithms used in linear algebra can be rewritten in such a way that they will be valid over any field and complete idempotent semiring (including semirings of intervals; see below and [58,59,90], where an interval version of the idempotent linear algebra and the corresponding universal algorithms are discussed).

As a rule, iterative algorithms (beginning with the successive approximation method) for solving differential equations (e.g., methods of Euler, Euler–Cauchy, Runge–Kutta, Adams, a number of important versions of the difference approximation method, and the like), methods for calculating elementary and some special functions based on the expansion in Taylor’s series and continuous fractions (Padé approximations) and others are independent of the computer representation of numbers.

Calculations on computers usually are based on a floating-point arithmetic with a mantissa of a fixed length; i.e., computations are performed with fixed accuracy. Broadly speaking, with this approach only the relative rounding error is fixed, which can lead to a drastic loss of accuracy and invalid results (e.g., when summing series and subtracting close numbers). On the other hand, this approach provides rather high speed of computations. Many important numerical algorithms are designed to use floating-point arithmetic (with fixed accuracy) and ensure the maximum computation speed. However, these algorithms are not universal. The above mentioned Gauss–Jacobi quadrature formulas, computation of elementary and special functions on the basis of the best polynomial or rational approximations or Padé–Chebyshev approximations, and some others belong to this type. Such algorithms use nontrivial constants specified with fixed accuracy.

Recently, problems of accuracy, reliability, and authenticity of computations (including the effect of rounding errors) have gained much attention; in part, this fact is related to the ever-increasing performance of computer hardware. When errors in initial data and rounding errors strongly affect

the computation results, such as in ill-posed problems, analysis of stability of solutions, etc., it is often useful to perform computations with improved and variable accuracy. In particular, the rational arithmetic, in which the rounding error is specified by the user [52], can be used for this purpose. This arithmetic is a useful complement to the interval analysis [70]. The corresponding computational algorithms must be universal (in the sense that they must be independent of the computer representation of numbers).

13. UNIVERSAL ALGORITHMS OF LINEAR ALGEBRA OVER SEMIRINGS

The most important linear algebra problem is to solve the system of linear equations

$$(24) \quad AX = B,$$

where A is a matrix with elements from the basic field and X and B are vectors (or matrices) with elements from the same field. It is required to find X if A and B are given. If A in (24) is not the identity matrix I , then system (24) can be written in form (23), i.e.,

$$(25) \quad X = AX + B.$$

It is well known that the form (25) is convenient for using the successive approximation method. Applying this method with the initial approximation $X_0 = 0$, we obtain the solution

$$(26) \quad X = A^*B,$$

where

$$(27) \quad A^* = I + A + A^2 + \cdots + A^n + \cdots$$

On the other hand, it is clear that

$$(28) \quad A^* = (I - A)^{-1},$$

if the matrix $I - A$ is invertible. The inverse matrix $(I - A)^{-1}$ can be considered as a regularized sum of the formal series (27).

The above considerations can be extended to a broad class of semirings.

The closure operation for matrix semirings $\text{Mat}_n(S)$ can be defined and computed in terms of the closure operation for S (see Subsection 11.3 above); some methods are described in [1, 2, 7, 14, 15, 26–29, 33, 37, 51, 59, 83, 84, 87]. One such method is described below (*LDM*-factorization), see [45].

If S is a field, then, by definition, $x^* = (1 - x)^{-1}$ for any $x \neq 1$. If S is an idempotent semiring, then, by definition,

$$(29) \quad x^* = \mathbf{1} \oplus x \oplus x^2 \oplus \cdots = \sup\{\mathbf{1}, x, x^2, \dots\},$$

if this supremum exists. Recall that it exists if S is complete, see section 4.2.

Consider a nontrivial universal algorithm applicable to matrices over semirings with the closure operation defined.

Example 65 (Semiring *LDM*-Factorization). Factorization of a matrix into the product $A = LDM$, where L and M are lower and upper triangular matrices with a unit diagonal, respectively, and D is a diagonal matrix, is used for solving matrix equations $AX = B$. We construct a similar decomposition for the Bellman equation $X = AX \oplus B$.

For the case $AX = B$, the decomposition $A = LDM$ induces the following decomposition of the initial equation:

$$(30) \quad LZ = B, \quad DY = Z, \quad MX = Y.$$

Hence, we have

$$(31) \quad A^{-1} = M^{-1}D^{-1}L^{-1},$$

if A is invertible. In essence, it is sufficient to find the matrices L , D and M , since the linear system (30) is easily solved by a combination of the forward substitution for Z , the trivial inversion of a diagonal matrix for Y , and the back substitution for X .

Using (30) as a pattern, we can write

$$(32) \quad Z = LZ \oplus B, \quad Y = DY \oplus Z, \quad X = MX \oplus Y.$$

Then

$$(33) \quad A^* = M^* D^* L^*.$$

A triple (L, D, M) consisting of a lower triangular, diagonal, and upper triangular matrices is called an *LDM-factorization* of a matrix A if relations (32) and (33) are satisfied. We note that in this case, the principal diagonals of L and M are zero.

The modification of the notion of *LDM-factorization* used in matrix analysis for the equation $AX = B$ is constructed in analogy with a construction suggested by Carré in [14, 15] for *LU-factorization*.

We stress that the algorithm described below can be applied to matrix computations over any semiring under the condition that the unary operation $a \mapsto a^*$ is applicable every time it is encountered in the computational process. Indeed, when constructing the algorithm, we use only the basic semiring operations of addition \oplus and multiplication \odot and the properties of associativity, commutativity of addition, and distributivity of multiplication over addition.

If A is a symmetric matrix over a semiring with a commutative multiplication, the amount of computations can be halved, since M and L are mapped into each other under transposition.

We begin with the case of a triangular matrix $A = L$ (or $A = M$). Then, finding X is reduced to the forward (or back) substitution.

Forward substitution

We are given:

- $L = \|l_j^i\|_{i,j=1}^n$, where $l_j^i = \mathbf{0}$ for $i \leq j$ (a lower triangular matrix with a zero diagonal);
- $B = \|b^i\|_{i=1}^n$.

It is required to find the solution $X = \|x^i\|_{i=1}^n$ to the equation $X = LX \oplus B$. The program fragment solving this problem is as follows.

```

for  $i = 1$  to  $n$  do
{    $x^i := b^i$ ;
  for  $j = 1$  to  $i - 1$  do
     $x^i := x^i \oplus (l_j^i \odot x^j)$ ; }

```

Back substitution

We are given

- $M = \|m_j^i\|_{i,j=1}^n$, where $m_j^i = \mathbf{0}$ for $i \geq j$ (an upper triangular matrix with a zero diagonal);
- $B = \|b^i\|_{i=1}^n$.

It is required to find the solution $X = \|x^i\|_{i=1}^n$ to the equation $X = MX \oplus B$. The program fragment solving this problem is as follows.

```

for  $i = n$  to 1 step  $-1$  do
{    $x^i := b^i$ ;
  for  $j = n$  to  $i + 1$  step  $-1$  do
     $x^i := x^i \oplus (m_j^i \odot x^i)$ ; }

```

Both algorithms require $(n^2 - n)/2$ operations \oplus and \odot .

Closure of a diagonal matrix

We are given

- $D = \text{diag}(d_1, \dots, d_n)$;
- $B = \|b^i\|_{i=1}^n$.

It is required to find the solution $X = \|x^i\|_{i=1}^n$ to the equation $X = DX \oplus B$. The program fragment solving this problem is as follows.

```

for  $i = 1$  to  $n$  do
   $x^i := (d_i)^* \odot b^i$ ;

```

This algorithm requires n operations $*$ and n multiplications \odot .

General case

We are given

- $L = \|l_j^i\|_{i,j=1}^n$, where $l_j^i = \mathbf{0}$ if $i \leq j$;
- $D = \text{diag}(d_1, \dots, d_n)$;
- $M = \|m_j^i\|_{i,j=1}^n$, where $m_j^i = \mathbf{0}$ if $i \geq j$;
- $B = \|b^i\|_{i=1}^n$.

It is required to find the solution $X = \|x^i\|_{i=1}^n$ to the equation $X = AX \oplus B$, where L , D , and M form the *LDM*-factorization of A . The program fragment solving this problem is as follows.

FORWARD SUBSTITUTION

for $i = 1$ to n do

{ $x^i := b^i$;

for $j = 1$ to $i - 1$ do

$x^i := x^i \oplus (l_j^i \odot x^j)$; }

CLOSURE OF A DIAGONAL MATRIX

for $i = 1$ to n do

$x^i := (d_i)^* \odot b^i$;

BACK SUBSTITUTION

for $i = n$ to 1 step -1 do

{ for $j = n$ to $i + 1$ step -1 do

$x^i := x^i \oplus (m_j^i \odot x^j)$; }

Note that x^i is not initialized in the course of the back substitution. The algorithm requires $n^2 - n$ operations \oplus , n^2 operations \odot , and n operations $*$.

LDM-factorization

We are given

- $A = \|a_j^i\|_{i,j=1}^n$.

It is required to find the *LDM*-factorization of A : $L = \|l_j^i\|_{i,j=1}^n$, $D = \text{diag}(d_1, \dots, d_n)$, and $M = \|m_j^i\|_{i,j=1}^n$, where $l_j^i = \mathbf{0}$ if $i \leq j$, and $m_j^i = \mathbf{0}$ if $i \geq j$.

The program uses the following internal variables:

- $C = \|c_j^i\|_{i,j=1}^n$;
- $V = \|v^i\|_{i=1}^n$;
- d .

INITIALISATION

for $i = 1$ to n do

for $j = 1$ to n do

$$c_j^i = a_j^i;$$

MAIN LOOP

for $j = 1$ to n do

{ for $i = 1$ to j do

$$v^i := a_j^i;$$

for $k = 1$ to $j - 1$ do

for $i = k + 1$ to j do

$$v^i := v^i \oplus (a_k^i \odot v^k);$$

for $i = 1$ to $j - 1$ do

$$a_j^i := (a_i^i)^* \odot v^i;$$

$$a_j^j := v^j;$$

for $k = 1$ to $j - 1$ do

for $i = j + 1$ to n do

$$a_j^i := a_j^i \oplus (a_k^i \odot v^k);$$

$$d = (v^j)^*;$$

for $i = j + 1$ to n do

$$a_j^i := a_j^i \odot d; \}$$

This algorithm requires $(2n^3 - 3n^2 + n)/6$ operations \oplus , $(2n^3 + 3n^2 - 5n)/6$ operations \odot , and $n(n+1)/2$ operations $*$. After its completion, the matrices L , D , and M are contained, respectively, in the lower triangle, on the diagonal, and in the upper triangle of the matrix C . In the case when A is symmetric about the principal diagonal and the semiring over which the matrix is defined is commutative, the algorithm can be modified in such a way that the number of operations is reduced approximately by a factor of two.

Other examples can be found in [14, 15, 26–29, 37, 38, 84, 87].

Note that to compute the matrices A^* and A^*B it is convenient to solve the Bellman equation (25).

Some other interesting and important problems of linear algebra over semirings are examined, e.g., in [9, 12, 13, 16, 23, 24, 26–29, 31, 75–77, 79, 98–101].

Remark 66. It is well known that linear problems and equations are especially convenient for parallelization, see, e.g., [97]. Standard methods (including the so-called block methods) constructed in the framework of the traditional mathematics can be extended to universal algorithms over semirings (the correspondence principle!). For example, formula (22) discussed in Subsection 11.3 leads to a simple block method for parallelization of the closure operations. Other standard methods of linear algebra [97] can be used in a similar way.

14. THE CORRESPONDENCE PRINCIPLE FOR COMPUTATIONS

Of course, the idempotent correspondence principle is valid for algorithms as well as for their software and hardware implementations [40, 42, 44, 51]. Thus:

If we have an important and interesting numerical algorithm, then there is a good chance that its semiring analogs are important and interesting as well.

In particular, according to the superposition principle, analogs of linear algebra algorithms are especially important. Note that numerical algorithms for standard infinite-dimensional linear problems over idempotent semirings (i.e., for problems related to idempotent integration, integral operators and transformations, the Hamilton-Jacobi and generalized Bellman equations) deal with the corresponding finite-dimensional (or finite) “linear approximations”. Nonlinear algorithms often can be approximated by linear ones. Thus the idempotent linear algebra is a basis for the idempotent numerical analysis.

Moreover, it is well-known that linear algebra algorithms easily lend themselves to parallel computation; their idempotent analogs admit parallelization as well. Thus we obtain a systematic way of applying parallel computing to optimization problems.

Basic algorithms of linear algebra (such as inner product of two vectors, matrix addition and multiplication, etc.) often do not depend on concrete semirings, as well as on the nature of domains containing the elements of vectors and matrices. Algorithms to construct the closure $A^* = I \oplus A \oplus A^2 \oplus \dots \oplus A^n \oplus \dots = \bigoplus_{n=1}^{\infty} A^n$ of an idempotent matrix A can be derived from standard methods for calculating $(I - A)^{-1}$. For the Gauss–Jordan elimination method (via LU-decomposition) this trick was used in [84], and the corresponding algorithm is universal and can be applied both to the Bellman equation and to computing the inverse of a real (or complex) matrix $(I - A)$. Computation of A^{-1} can be derived from this universal algorithm with some obvious cosmetic transformations.

Thus it seems reasonable to develop universal algorithms that can deal equally well with initial data of different domains sharing the same basic structure [40, 42, 44].

15. THE CORRESPONDENCE PRINCIPLE FOR HARDWARE DESIGN

A systematic application of the correspondence principle to computer calculations leads to a unifying approach to software and hardware design.

The most important and standard numerical algorithms have many hardware realizations in the form of technical devices or special processors. *These devices often can be used as prototypes for new hardware units generated by substitution of the usual arithmetic operations for its semiring analogs and by addition tools for performing neutral elements $\mathbf{0}$ and $\mathbf{1}$* (the latter usually is not difficult). Of course, the case of numerical semirings consisting of real numbers (maybe except neutral elements) and semirings of numerical intervals is the most simple and natural [39, 40, 42–44, 51, 58, 59, 90]. Note that for semifields (including \mathbf{R}_{\max} and \mathbf{R}_{\min}) the operation of division is also defined.

Good and efficient technical ideas and decisions can be transferred from prototypes to new hardware units. Thus the correspondence principle generated a regular heuristic method for hardware design. Note that to get a patent it is necessary to present the so-called ‘invention formula’, that is to indicate a prototype for the suggested device and the difference between these devices.

Consider (as a typical example) the most popular and important algorithm of computing the scalar product of two vectors:

$$(34) \quad (x, y) = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

The universal version of (34) for any semiring A is obvious:

$$(35) \quad (x, y) = (x_1 \odot y_1) \oplus (x_2 \odot y_2) \oplus \cdots \oplus (x_n \odot y_n).$$

In the case $A = \mathbf{R}_{\max}$ this formula turns into the following one:

$$(36) \quad (x, y) = \max\{x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n\}.$$

This calculation is standard for many optimization algorithms, so it is useful to construct a hardware unit for computing (36). There are many

different devices (and patents) for computing (34) and every such device can be used as a prototype to construct a new device for computing (36) and even (35). Many processors for matrix multiplication and for other algorithms of linear algebra are based on computing scalar products and on the corresponding “elementary” devices respectively, etc.

There are some methods to make these new devices more universal than their prototypes. There is a modest collection of possible operations for standard numerical semirings: max, min, and the usual arithmetic operations. So, it is easy to construct programmable hardware processors with variable basic operations. Using modern technologies it is possible to construct cheap special-purpose multi-processor chips implementing examined algorithms. The so-called systolic processors are especially convenient for this purpose. A systolic array is a ‘homogeneous’ computing medium consisting of elementary processors, where the general scheme and processor connections are simple and regular. Every elementary processor pumps data in and out performing elementary operations in a such way that the corresponding data flow is kept up in the computing medium; there is an analogy with the blood circulation and this is a reason for the term “systolic”, see e.g., [40, 42, 44, 45, 66, 83, 84, 87].

Some systolic processors for the general algebraic path problem are presented in [83, 84, 87]. In particular, there is a systolic array of $n(n + 1)$ elementary processors which performs computations of the Gauss–Jordan elimination algorithm and can solve the algebraic path problem within $5n - 2$ time steps. Of course, hardware implementations for important and popular basic algorithms increase the speed of data processing.

The so-called GPGPU (General-Purpose computing on Graphics Processing Units) technique is another important field for applications of the correspondence principle. The matter is that graphic processing units (hidden in modern laptop and desktop computers) are potentially powerful processors for solving numerical problems. The recent tremendous

progress in graphical processing hardware and software resulted in new “open” programmable parallel computational devices (special processors), see, e.g., [11,78,102]. These devices are going to be standard for coming PC (personal computers) generations. Initially used for graphical processing only (at that time they were called GPU), today they are used for various fields, including audio and video processing, computer simulation, and encryption. But this list can be considerably enlarged following the correspondence principle: the basic operations would be used as parameters. Using the technique described in this paper (see also our references), standard linear algebra algorithms can be used for solving different problems in different areas. In fact, the hardware supports all operations needed for the most important idempotent semirings: plus, times, min, max. The most popular linear algebra packages [ATLAS (Automatically Tuned Linear Algebra Software), LAPACK, PLASMA (Parallel Linear Algebra for Scalable Multicore Architectures)] can already use GPGPU, see [103–105]. We propose to make these tools more powerful by using parameterized algorithms.

Linear algebra over the most important numerical semirings generates solutions for many concrete problems in different areas, see above.

Note that to be consistent with operations we have to redefine zero (0) and unit (1) elements (see above); comparison operations must be also redefined as it is described above. Once the operations are redefined, then the most of basic linear algebra algorithms, including back and forward substitution, Gauss elimination method, Jordan elimination method and others could be rewritten for new domains and data structures. Combined with the power of the new parallel hardware this approach could change PC from entertainment devices to power full instruments.

16. THE CORRESPONDENCE PRINCIPLE FOR SOFTWARE DESIGN

Software implementations for universal semiring algorithms are not as efficient as hardware ones (with respect to the computation speed) but they are much more flexible. Program modules can deal with abstract (and variable) operations and data types. These operations and data types can be defined by the corresponding input data. In this case they can be generated by means of additional program modules. For programs written in this manner it is convenient to use special techniques of the so-called object oriented (and functional) design, see, e.g., [60,80,91]. Fortunately, powerful tools supporting the object-oriented software design have recently appeared including compilers for real and convenient programming languages (e.g. C^{++} and Java) and modern computer algebra systems.

Recently, this type of programming technique has been dubbed generic programming (see, e.g., [8, 80]). To help automate the generic programming, the so-called Standard Template Library (STL) was developed in the framework of C^{++} [80,91]. However, high-level tools, such as STL, possess both obvious advantages and some disadvantages and must be used with caution.

It seems that it is natural to obtain an implementation of the correspondence principle approach to scientific calculations in the form of a powerful software system based on a collection of universal algorithms. This approach ensures a working time reduction for programmers and users because of the software unification. The arbitrary necessary accuracy and safety of numeric calculations can be ensured as well.

This software system may be especially useful for designers of algorithms, software engineers, students and mathematicians.

Note that there are some software systems oriented to calculations with idempotent semirings like \mathbf{R}_{\max} ; see, e.g., [82]. However these systems do not support universal algorithms.

17. INTERVAL ANALYSIS IN IDEMPOTENT MATHEMATICS

Traditional interval analysis is a nontrivial and popular mathematical area, see, e.g., [4, 24, 35, 70, 74, 77]. An “idempotent” version of interval analysis (and moreover interval analysis over positive semirings) appeared in [58, 59, 90]. Later the idempotent interval analysis has attracted many experts in tropical linear algebra and applications, see, e.g., [16, 24, 31, 75, 76, 101]. We also mention the closely related interval analysis over the positive semiring \mathbf{R}_+ discussed in [9].

Let a set S be partially ordered by a relation \preceq . A *closed interval* in S is a subset of the form $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{x \in S \mid \underline{\mathbf{x}} \preceq x \preceq \overline{\mathbf{x}}\}$, where the elements $\underline{\mathbf{x}} \preceq \overline{\mathbf{x}}$ are called *lower* and *upper bounds* of the interval \mathbf{x} . The order \preceq induces a partial ordering on the set of all closed intervals in S : $\mathbf{x} \preceq \mathbf{y}$ iff $\underline{\mathbf{x}} \preceq \underline{\mathbf{y}}$ and $\overline{\mathbf{x}} \preceq \overline{\mathbf{y}}$.

A *weak interval extension* $I(S)$ of an ordered semiring S is the set of all closed intervals in S endowed with operations \oplus and \odot defined as $\mathbf{x} \oplus \mathbf{y} = [\underline{\mathbf{x}} \oplus \underline{\mathbf{y}}, \overline{\mathbf{x}} \oplus \overline{\mathbf{y}}]$, $\mathbf{x} \odot \mathbf{y} = [\underline{\mathbf{x}} \odot \underline{\mathbf{y}}, \overline{\mathbf{x}} \odot \overline{\mathbf{y}}]$ and a partial order induced by the order in S . The closure operation in $I(S)$ is defined by $\mathbf{x}^* = [\underline{\mathbf{x}}^*, \overline{\mathbf{x}}^*]$. There are some other interval extensions (including the so-called strong interval extension [59]) but the weak extension is more convenient.

The extension $I(S)$ is idempotent if S is an idempotent semiring. A universal algorithm over S can be applied to $I(S)$ and we shall get an interval version of the initial algorithm. Usually both the versions have the same complexity. For the discrete stationary Bellman equation and the corresponding optimization problems on graphs, interval analysis was examined in [58, 59] in details. Other problems of idempotent linear algebra were examined in [16, 24, 31, 75, 76].

Idempotent mathematics appears to be remarkably simpler than its traditional analog. For example, in traditional interval arithmetic, multiplication of intervals is not distributive with respect to addition of intervals, whereas in idempotent interval arithmetic this distributivity is preserved.

Moreover, in traditional interval analysis the set of all square interval matrices of a given order does not form even a semigroup with respect to matrix multiplication: this operation is not associative since distributivity is lost in the traditional interval arithmetic. On the contrary, in the idempotent (and positive) case associativity is preserved. Finally, in traditional interval analysis some problems of linear algebra, such as solution of a linear system of interval equations, can be very difficult (more precisely, they are NP -hard, see [19, 24, 35, 36] and references therein). It was noticed in [58, 59] that in the idempotent case solving an interval linear system requires a polynomial number of operations (similarly to the usual Gauss elimination algorithm). The remarkable simplicity of idempotent interval arithmetic is due to the following properties: the monotonicity of arithmetic operations and the positivity of all elements of an idempotent semiring.

Interval estimates in idempotent mathematics are usually exact. In the traditional theory such estimates tend to be overly pessimistic.

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